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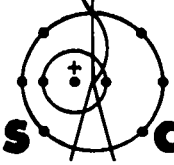

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THEORY OF THE SCATTERING OF PIONS BY NUCLEAR MATTER

by

H. A. Bethe and Mikkel B. Johnson

ABSTRACT

We derive a relation between the (complex) wave number k of a pion in nuclear matter of density ρ and its energy ω. The relation is determined by the pion self-energy $\Pi(\omega,\rho)$. Following Dover and Lemmer, Π is an integral of the forward scattering amplitude $f(k,\omega,\rho)$ over the density. A formalism analagous to Chew and Low is used to find f in nuclear matter; the result is similar to the Chew-Low theory for free nucleons. Pauli principle and finite mass of the nucleon can be taken into account easily, and give a significant modification. The resulting formulae are evaluated in an approximate model. The wave number k stays moderate at all energies and densities, in contrast to most of the earlier theories.

The resonance is defined as the energy at which the real part of the forward scattering from a nucleus such as ^{12}C is zero. Theory predicts that this energy is 6 MeV lower than the free-nucleon resonance; experimentally the downward shift is 20 ± 20 MeV. The theory is applied to finite nuclei, using a local-density approximation with a simple correction at the nuclear surface. Elastic scattering near resonance is governed by the low-density surface region of the nucleus (density 0.2 to 0.3 of central density); this explains the success of earlier theories. Inelastic scattering is dominated by the large refractive index for pions (up to 2); inelastically scattered pions will either come out at very low energy or be captured by the nucleus.

I. HISTORICAL INTRODUCTION

The problem of the interaction of pions with nuclei has received attention for at least 15 years. Many good experiments have been done, and soon the meson facilities at Los Alamos and elsewhere will produce further, more accurate and more detailed experimental information. A renewed theoretical study is in order.

The most influential theory has been that of Kisslinger.[1] It can be based on the well-known formula[2] of physical optics

$$k^2 = k_0^2 + 4\pi\rho f(k,0) \tag{1.1}$$

in which k_0 is the wave number in free space,

$$k_0^2 = \omega^2 - \mu^2 \tag{1.2}$$

(ω = energy, μ = mass of pion, $\hbar = c = 1$), k is the wave number in the medium (nuclear matter), ρ the density[3] of nucleons, and $f(k,0)$ the forward scattering amplitude for the scattering of a pion by one nucleon[4] in nuclear matter. Since the main interaction between pion and nucleon is the p-state

interaction

$$g \vec{\sigma} \cdot \nabla \phi \qquad (1.3)$$

($\vec{\sigma}$ the nucleon spin, ϕ the pion wave function), we may assume that

$$f(k,0) = a_0(\omega) + a_1(\omega)k^2 \quad , \qquad (1.4)$$

where a_0 represents the s-state and a_1 the p-state scattering. Kisslinger now assumes that the energy-dependent factors a_0 and a_1 can be deduced from the scattering of pions by free nucleons. It is this assumption which we shall change in this report.

Inserting Eq. (1.4) into Eq. (1.1), we obtain

$$k^2 = k_0^2 + 4\pi\rho a_0(\omega) + 4\pi\rho a_1(\omega)k^2 \qquad (1.5)$$

and

$$k^2 = \frac{\omega^2 - \mu^2 + 4\pi\rho a_0(\omega)}{1 - 4\pi\rho a_1(\omega)} \quad . \qquad (1.6)$$

The quantity $a_1(\omega)$ has been carefully evaluated by Ericson and Ericson[5] at the threshold $\omega = \mu$, using the best available numbers on the scattering by free nucleons. This yields

$$a_1(\mu) = (1 + \mu/M)^{-1} c_0 = 0.198\ \mu^{-3} \qquad (1.7)$$

(c_0 is the Ericsons' notation, M = nucleon mass). This means that the denominator in Eq. (1.6) vanishes for

$$\rho = \rho_1 = 0.40\ \mu^3 \quad , \qquad (1.8)$$

which is less than the density of nuclear matter,[6]

$$\rho_0 = 0.16\ \text{fm}^{-3} = 0.47\ \mu^3 \quad . \qquad (1.9)$$

Assuming the numerator of Eq. (1.6) to be positive which it will certainly be at higher energy,[7] the wave number k turns out to be purely imaginary (!), and at the "critical" density ρ_1, it is infinite. We shall call this the Kisslinger syndrome, and we consider it serious trouble.

Kisslinger has pointed out that for energies $\omega > \mu$, the free-nucleon scattering $a_1(\omega)$ is complex so that the k^2 from Eq. (1.6) does not actually become infinite at any ρ. But it does become very large, and this gives rise to unphysical behavior of the pion wave function and to wrong conclusions[8] on such matters as the one-nucleon absorption of pions by the process

$$N + \pi = N' + \gamma \qquad (1.10)$$

(N = nucleon, N' = nucleon of different charge). At $\omega \approx \mu$, the free-nucleon scattering $a_1(\mu)$ is purely real because the phase shifts are small, but the Ericsons have pointed out that an important imaginary term is contributed by the two-nucleon absorption

$$N + N + \pi = N' + N + \text{kinetic energy} \qquad (1.11)$$

and have calculated this contribution.[5] It also persists for $\omega > \mu$. The Ericsons have found that the Kisslinger syndrome is further mitigated by the Lorentz-Lorenz correction[5,2] (see Sec. II).

The Ericsons and others[9,10] have applied their theory to an understanding of the shift and width of the energy levels of pionic atoms, and have achieved impressive agreement with experiment. Their results remain essentially unchanged in our theory.

In his first paper, Kisslinger extended his theory to finite nuclei in which $\rho = \rho(\vec{r})$ is a function of position. The last term in Eq. (1.5) is made into a self-adjoint operator by replacing it by[11]

$$\begin{aligned} &-4\pi a_1(\omega) \quad \nabla\cdot(\rho\nabla) \\ &= -4\pi a_1(\omega)\ (\rho\nabla^2 + \nabla\rho\cdot\nabla) \quad . \end{aligned} \qquad (1.12)$$

This theory has been used widely[12] to calculate the scattering of pions around 80 MeV by ^{12}C and other nuclei; good agreement with the differential cross section has been obtained. At 30 MeV, Marshall et al.[13] were also able to fit their experimental data with the Kisslinger scheme, but they needed a much larger repulsive s-wave scattering than the free-nucleon scattering data indicate. Their b_0 is -4, whereas the theoretical value is -0.5.

In an effort to cover energies closer to the 33-resonance, Ericson and Hüfner[14] represent $f(k,0)$ by a Breit-Wigner formula. For the coupling of the p-state pion to the nucleon, they alternatively use k^2 [as we do in Eq. (1.4)] or k_0^2, the free-pion wave number. In the first case, they find that the resonance is shifted down to lower energies, and in the second case, it is shifted slightly up; comparison with experiment is rather inconclusive. For the width of the resonance, they take the values deduced from free nucleon-pion scattering. The width is then a function of the pion energy only, $\Gamma(E)$ or $\Gamma(k_0)$. We do not believe this is justified; Γ should depend on k, the pion wave number in nuclear matter (see below, where the work of Barshay et al. is discussed).

Dover and Lemmer[15] go back to first principles. They show that k in nuclear matter is given by the self-energy of the pion [our Eq. (2.2)]. Then they establish that the self-energy can be calculated from the scattering of the pion by single nucleons. For the scattering, they develop a theory analagous to that of Chew and Low[16] for the scattering by a free nucleon. The paper by Dover and Lemmer is done with great care, and we shall use it as the basis of our general theory in Sec. II. Unfortunately, in the further evaluation of the scattering amplitude, Dover and Lemmer make the same assumptions as Kisslinger, i.e., k^2 is used in the coupling of pion and nucleon, as in Eq. (1.4), but in evaluating the equivalent of $a_1(\omega)$ in Eq. (1.4), they explicitly set $\vec{k} = \vec{k}_0$, the incident momentum [Ref. 15, text below Eq. (3.30)]. This is incorrect, since Eq. (1.1) shows that $k \neq k_0$; in fact, as we show in Sec. VII, this effect is much larger than the effects Dover and Lemmer take into account, such as the Pauli principle. The Pauli principle was also considered by Eisenberg and Weber.[17]

The first authors to give the correct treatment of the resonance were Barshay, Rostokin, and Vagradov.[18] They introduce explicitly the Δ-particle (33-resonance) as a separate particle. Absorption of a pion may transform a nucleon into a Δ, which then changes back into a nucleon when the pion is reemitted. The coupling constant πNΔ is known from the width of the "free" Δ resonance. In nuclear matter the effective width is automatically proportional to k^3 (Ref. 18b, p. 200, top); the authors get a reasonable shape of the total and the reaction cross section.

An especially simple treatment of this Δ-model was given by Barshay, Brown, and Rho.[19] They consider that, after absorption or emission of a pion, a nucleon may either remain a nucleon or transform into a Δ (Ref. 19, Fig. 2). If the pion has very low momentum, $k \ll p_F$ (the Fermi momentum of the nucleons), the first process, $N + \pi \to N$, is forbidden by the Pauli principle, but the second process, $N + \pi \to \Delta$, remains unaffected. This picture gives just the correct behavior of the scattering amplitude f for very low k, which is important for the interpretation of the energy of pionic atoms (Sec. VI). The authors also show that an earlier paper of one of the present authors[20] postulated an excessive effect of the Pauli principle for low k.

The introduction of the Δ, while giving a very simple description of many phenomena, raises some new questions, in particular how does the Δ interact with nuclear matter? Barshay, Rostokin, and Vagradov[18] explicitly introduce the potential energy V_Δ of the Δ in nuclear matter, and find that the self-energy of the pion is sensitive to V_Δ. It seems difficult to calculate V_Δ; for instance, it should contain the rather large increase of the self-energy pointed out by Sawyer[21] which is due to the fact that the decay of the Δ into nucleon states inside the Fermi sea is forbidden. Because of the uncertainty in V_Δ, and other reasons, we felt justified in continuing our calculations, begun in the summer of 1972, which are based on Chew-Low theory; after all that theory generates the Δ. But we agree with Barshay et al. that the effective width of the resonance is proportional to k^3, with k the momentum <u>in</u> the medium.

The use of k^3 not only increases the width of the resonance, in agreement with observation, but also keeps down the resonance cross section of the scattering by one nucleon inside nuclear matter, σ_{rN}. This is essential because this cross section should not exceed (or at least not by much) πr_0^2 where $r_0 A^{1/3}$ is the nuclear radius. Otherwise, the scattering cross sections of neighboring nucleons would overlap, which is very unreasonable. Having k^2 in Eq. (1.4), and $a_1(\omega)$ the same as for <u>free</u> pions makes σ_{rN} very large, in fact even larger than σ_r for free pions which is 130 mb, while $\pi r_0^2 \approx 40$ mb.

Many alternative approaches have been published, far too many to be enumerated. Landau, Phatak, and Tabakin[22] use a separable potential between nucleon and pion to treat off-shell propagation of the pion. Off-shell propagation is, of course, already treated very well by the Chew-Low method[16] in which dependence of the scattering matrix on k and on ω are cleanly separated; the Chew-Low method stays more closely in the framework of standard pion-nucleon scattering theory.

Schmit and Dedonder[23] have also properly emphasized the importance of off-shell propagation of the pion in nuclear matter. In applying his theory, Dedonder points out that the vanishing of the real part of the scattering amplitude is a good criterion for the position of the resonance (see Sec. VII) and finds that his theory gives better values for this position than earlier ones.

Gibbs[24] uses a completely different approach. He does not determine an optical potential (or self-energy, or wave number) of the pion in nuclear matter, but considers the successive scatterings of the pion by individual nucleons. This involves, of course, a prodigious amount of computing, but Gibbs has simplified the problem sufficiently to make it manageable, at least for a light nucleus like ^{12}C. We believe that our nuclear matter approach can simplify the scattering problem and bring to light the essential physics with much less computing. In any case, for heavy nuclei, the approach of Gibbs becomes completely unmanageable. Although, in principle, the method of Gibbs requires knowledge of the off-shell T-matrix, he has invoked approximations which render results insensitive to the off-shell extrapolation.

Although the need for going "off energy shell" has been recognized, a tractable calculational procedure, properly motivated by physical considerations, has remained elusive. From our point of view, Eisenberg[25] has recently taken a step in the correct direction. He considers the interaction of the transmitted pion (q in Fig. 1, Diagram 4u) with another nucleon. This is the first approximation to using a refractive index for pions, as in Eqs. (1.1) or (2.2). In agreement with our later results, the most important effect of the interaction of q is the addition of an imaginary self-energy.

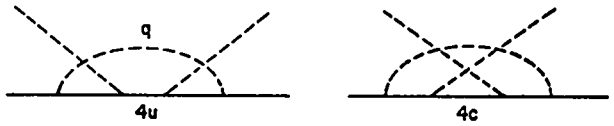

Fig. 1. Two fourth-order diagrams for the scattering of a pion (dashed) by a nucleon (solid). Notation due to Dover and Lemmer.[15]

II. GENERAL THEORY

The fundamental theory has been very well represented by Dover and Lemmer.[15] They consider the propagator of a pion in nuclear matter [their Eq. (2.8)],

$$D(k,\omega) = \left[\mu^2 + k^2 - \omega^2 - \Pi(k,\omega)\right]^{-1} \quad . \qquad (2.1)$$

where $\Pi(k,\omega)$ is the pion self-energy[26] in nuclear matter. This is, of course, complex (a) because the pion of momentum $\vec{k}$ may be scattered into other directions (which corresponds to a decay of state $\vec{k}$), (b) because the pion may be absorbed by one or two nucleons [Eqs. (1.10) and (1.11)], and (c) because it may produce further pions by collision with nucleons. Process (c) becomes important only well above the 33-resonance, so we shall omit it. Process (b) will be considered in Sec. VI. For the present, we shall only consider scattering, process (a).

The pole of the propagator Eq. (2.1) defines the energy-momentum relation in the nuclear medium,

$$k^2 = k_0^2 + \Pi(k,\omega) \quad , \qquad (2.2)$$

which has a similar form to Eq. (1.1). The self-energy diagrams can be divided into those involving only one nucleon and those involving several. Some of the one-nucleon diagrams are given in Ref. 15, Fig. 7. A typical two-nucleon diagram is shown in our Fig. 2. The pion is absorbed by nucleon a which goes from an occupied state p_a to a normally unoccupied one, p_u. Then it interacts with nucleon b, which then finally re-emits the pion. The interaction must not be transmitted by a pion because then Fig. 2 would be reducible to two scatterings by single nucleons, but any heavier meson, e.g., σ, ρ, or ω is possible. So the interaction represents close correlation between the nucleons a and b; we discuss its effect at the end of this section.

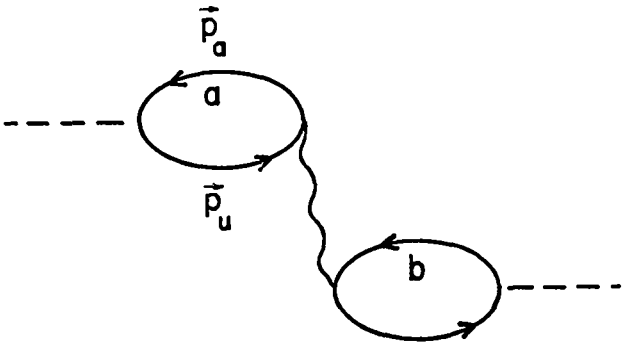

Fig. 2. A two-nucleon contribution to the pion self-energy; a and b are the two nucleons.

For the one-nucleon diagrams, Dover and Lemmer have shown that they are closely related to pion-nucleon scattering diagrams (their Fig. 8). They show [their Eq. (3.16)], that the functional derivative

$$\frac{\delta\Pi(k)}{\delta G(p)} = 4\pi i\, f(p\ k;\ p\ k) \quad , \tag{2.3}$$

where G(p) is the renormalized nucleon propagator[28] in the medium, and f is the scattering amplitude as a function of the four vectors p and k. The nucleon propagator G has a simple form if we use the one-nucleon approximation of nuclear matter theory.[29,30] We have then, exactly as shown in Ref. 15, Eq. (3.19),

$$G(\vec{p},p_0) = \frac{n(\vec{p})}{\varepsilon(\vec{p}) - p_0 + i\delta} + \frac{1 - n(\vec{p})}{\varepsilon(\vec{p}) - p_0 - i\delta} \quad , \tag{2.4}$$

where $n(\vec{p})$ is the occupation number of nucleon state $\vec{p}$, and where

$$\varepsilon(\vec{p}) = p^2/2M + U(\vec{p}) \tag{2.5}$$

with $U(\vec{p})$ the potential energy of a nucleon $\vec{p}$ in the field of all the others.[30] $U(\vec{p})$ takes into account the interaction of a nucleon with another in which both nucleons return to their original state, indicated by the wavy lines in Figs. 3a and 3b. These are in contrast to the interaction in Fig. 2 in which the excitation (provided originally by the pion) is transferred to another nucleon.

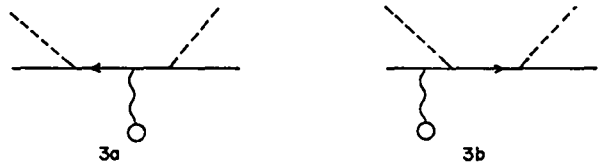

Fig. 3. Self-energy of a nucleon, (a) in a particle, (b) in a hole state.

In general, the pion self-energy will depend on all the occupation numbers $n(\vec{p})$ of the nucleon states. Dover and Lemmer[15] show, in their Eq. (3.21), that

$$\frac{\delta\Pi(k)}{\delta n(\vec{p})} = 4\pi\ f(k\ p,\ k\ p)\big|_{p_0=\varepsilon(p)} \quad , \tag{2.6}$$

where the last symbol simply indicates that all nucleons have to be taken on their energy shell; we shall imply this fact in the following.

We are interested in nuclear matter. In this case, all nucleon states inside the Fermi sea are occupied,

$$n(\vec{p}) = 1 \quad \text{if} \quad |\vec{p}| \leq p_F \quad . \tag{2.7}$$

In this case, the self-energy Π can depend only on the nuclear density

$$\rho = 4\int n(\vec{p})\,\frac{d^3p}{(2\pi)^3} = \frac{2}{3\pi^2}\,p_F^3 \tag{2.8}$$

(factor 4 for spin and isospin). As the density is increased from ρ to $\rho + d\rho$, one nucleon is added in each state in a spherical shell in momentum space with momentum $|\vec{p}| = p_F$. Therefore

$$\frac{d\Pi}{d\rho} = \int d\Omega_{\vec{p}}\ f(\vec{p}\ k,\ \vec{p}\ k)_{p=p_F} \quad , \tag{2.9}$$

$$\equiv 4\pi f_{Av}(p_F k) \quad . \tag{2.10}$$

The integral in Eq. (2.9) is over all <u>directions</u> of $\vec{p}$ while the magnitude[31] of p remains fixed at p_F. The scattered amplitude f in Eq. (2.10) must, of course, be calculated in nuclear matter, at the density ρ corresponding to p_F [Eq. (2.8)]. In doing

this, we must use the appropriate relation between $|\vec{k}|$ and ω of the pion, viz., that given by Eqs. (2.2) and (2.9). This will be done in Sec. III. Obviously, we have a self-consistency problem: f in Eq. (2.9) depends on the relation between $\vec{k}$ and ω. This relation is determined by $\Pi(\rho,k)$, and Π, in turn, depends on f.

It is interesting that, for any given density, f needs to be calculated only $p = p_F$. This clearly minimizes[32] the effect of the Pauli principle which is discussed in Sec. IV. It also ensures that this effect is given by an analytic function [see the discussion below Eq. (4.13)].

Using the average scattering amplitude defined in Eq. (2.10), we get

$$\Pi(k) = 4\pi \int f_{Av}(p_F, k)\, d\rho \quad . \tag{2.11}$$

If f_{Av} is independent of p_F, i.e., of density ρ, and only then, Eq. (2.11) leads to the simple, intuitive formula, Eq. (1.1). As is well known, that formula can be derived by considering the wavelets scattered by individual nucleons, which interfere constructively in the forward direction. From the theory of Dover and Lemmer which we have followed here, it is clear that Eq. (1.1) is only a low-density approximation; but it is satisfactory that the intricate Dover-Lemmer theory leads back to Eq. (1.1) at least in the low-density limit.

We now return to the contributions of two-nucleon diagrams (Fig. 2) to the pion self-energy. These have been treated simply and elegantly by Barshay, Brown, and Rho[19] whose general method was described in Sec. I. They show that the main effect of these diagrams is to give a Lorentz-Lorenz correction which expresses the fact that two nucleons can not be close together because of their strong short-range repulsion. This effect exists even if the repulsion has very short range (let us say, compared to the average distance between nucleons).

Earlier, the Lorentz-Lorenz correction was derived by the Ericsons[5] who used the analogy between the p-state pion interaction and electrostatics. They, as well as Barshay et al., find that the last term in Eq. (1.5) is replaced by

$$\frac{4\pi\rho a_1(\omega)}{1 + (4\pi/3)\rho a_1} k^2 \quad . \tag{2.12}$$

This means that the Kisslinger syndrome is mitigated but not eliminated. The density at which k^2 in Eq. (1.6) becomes infinite is now raised to

$$\rho_2 = (3/2)\ \rho_1 = 0.60\ \mu^3 \quad , \tag{2.13}$$

slightly above nuclear matter density [Eq. (1.9)]. But very large k^2 will persist.

There are other effects of two-nucleon correlations but these are probably smaller. Three-nucleon and higher correlations should be negligible.

The short-range correlation between nucleons, due to the repulsive forces, is not the only one that exists. There is also the long-range correlation due to Fermi statistics, and the medium range one, due to the attractive forces. The latter is very weak[33] and can be neglected. The Fermi correlation is taken into account by introducing the Pauli principle explicitly in the scattering by a single nucleon, as done by Dover and Lemmer and by us in Sec. IV.

III. SCATTERING BY A NUCLEON

We shall treat the scattering of a pion by a nucleon in nuclear matter by the same method as that used by Chew and Low[16] for the scattering by a free nucleon. This method has also been used for nuclear matter by Dover and Lemmer[15] whose work we have already used in Sec. II for the fundamental theory. Dover and Lemmer rederive the Chew-Low theory from Feynman diagrams and then introduce the Pauli principle for the nucleons. Unfortunately, they then make the approximation [below their Eq. (3.30)] that the momentum inside the nucleus $\vec{k}$ is the same as the outside momentum $\vec{k}_0$. We have already discussed in Sec. I that this is not legitimate, and we shall see later that the difference $k - k_0$ has very large effects.

In using the Chew-Low theory, we shall also use their one-meson approximation[34] which is by far the simplest way to take unitarity into account. This restriction to the "one-meson approximation" of Chew and Low may, to some extent, be justified by the

modern analysis of π-nucleon scattering. According to Donnachie and Hamilton,[35] over 75% of the strength of the Δ resonance is contributed by the Chew-Low theory, called "nucleon exchange" by Donnachie and Hamilton. Most of the rest is due to exchange of a σ-meson, more precisely of two pions correlated in a state of isospin and angular momentum 0 (Fig. 4). Only 2% is attributed to N^* exchange, which is perhaps the closest analog to the Chew-Low diagrams going beyond the one-meson approximation, and 0.6% is due to ρ-exchange.

It would, of course, be best to consider the Donnachie-Hamilton diagram (Fig. 4) also in nuclear matter. This would, however, make it necessary to consider the behavior of the σ-meson in nuclear matter. We have not felt able to do this, and have therefore confined ourselves to imitating the Chew-Low theory.

For the purposes of this section and the next, we shall assume that the nucleon has infinite mass, so that the pion energy remains unchanged in the scattering; thus ω and k have definite values. The energy lost by the pion to the nucleon will be treated in Sec. V.

The theory of this section applies to the scattering by a nucleon of <u>any</u> momentum $\vec{p}$. In the next section, when we discuss the Pauli principle, we shall make use of the fact, discussed below Eq. (2.9), that we need only the case $p = p_F$.

To have a theory of the Chew-Low type, it is necessary (and sufficient) to have a definite relation between energy ω and momentum k. (The momentum serves both to count the number of intermediate states, and to establish the unitarity relation, see Eq. (3.6).) For the scattering by <u>free</u> nucleons, the relation between ω and k is

$$\omega_{k_0} = (k^2 + \mu^2)^{\frac{1}{2}} \quad . \tag{3.1}$$

In nuclear matter this relation is changed because of Eq. (2.2). But for <u>fixed nuclear matter density</u> ρ, there is still a <u>definite</u> relation. Now, it is well known that the forward scattered amplitude $f(\vec{pk},\vec{pk})$ is complex. In fact, because of the optical theorem[36] the imaginary part of f is always positive. Hence, using Eqs. (2.2) and (2.11) for any <u>real</u> value of k the corresponding value of the energy ω_k has a negative imaginary part,

$$\text{Im } \omega_k < 0 \qquad (k \text{ real}) \quad . \tag{3.2}$$

Similarly, for any real ω, the corresponding wave number k_ω is such that

$$\text{Im } k_\omega > 0 \qquad (\omega \text{ real}) \quad . \tag{3.3}$$

Following precisely the procedure of Chew and Low, we define the amplitude $t_{kq}(z)$,

$$t_{kq}(z) = - v(k)\, v(q)\, 4\pi (4\omega_k \omega_q)^{-\frac{1}{2}} \times \times \sum_{\alpha=1}^{4} P_\alpha(k,q) h_\alpha(z) \quad , \tag{3.4}$$

where k is initial and q final momentum, z a complex variable replacing ω, v(k) a form factor going to zero as $k \to \infty$, $\alpha = (2I, 2J)$ distinguishes the partial waves, I is the total isospin and J the total angular momentum, and the P_α are projection operators of which we shall only need

$$P_{33} = (\delta_{kq} - 1/3\, \tau_q \tau_k^{\dagger})\, \mu^{-2} \left[3\, \vec{k} \cdot \vec{q} - (\vec{\sigma} \cdot \vec{k})(\vec{\sigma} \cdot \vec{q})\right] \quad . \tag{3.5}$$

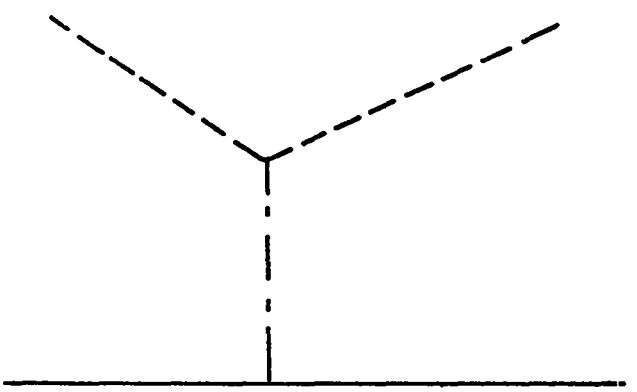

Fig. 4. The only important pion-nucleon interaction other than Chew-Low, according to Donnachie and Hamilton.[35] The line —•—•— denotes a σ-meson.

The important function is $h_\alpha(z)$. Chew and Low show that $h_{13} = h_{31}$ and therefore consider just the three functions $h_1 = h_{11}$, $h_2 = h_{13} = h_{31}$ and $h_3 = h_{33}$; only h_3 is important for us.

An essential part of the theory is the unitarity condition. To facilitate writing this, Chew and Low introduce the one-meson approximation, i.e., they assume that only a single meson can ever be present in any intermediate state (see the beginning of this section). Using this approximation, the unitarity condition becomes simply a relation between $h_\alpha(z)$ and phase shifts [Ref. 16, Eq. (34)].

To define phase shifts, it is essential that we assume the wave number k to be real. Only in this case does there exist the possibility of a Rayleigh scattering formula,

$$f(k,\theta) = k^{-1}\Sigma(2\ell + 1) \sin \delta_\ell e^{i\delta_\ell} P_\ell(\cos \theta) . \qquad (3.6)$$

As will be remembered in deriving this formula, one considers in- and out-going spherical waves, $e^{\pm ikr}/r$, and either one or the other of these will become nonsensical if k is complex.

The Chew-Low theory which uses the complex ω-plane gives us just enough freedom to choose k to be real. There are branch points at $\omega = \pm \mu$ in the Chew-Low theory because of the relation in Eq. (3.1), and these persist also in nuclear matter, as discussed in more detail in Sec. VI. A branch cut therefore has to start from $\omega = \mu$. Chew and Low choose this to lie along the real ω-axis which in their case is also the real k-axis. We choose it (for the present) to lie along the line of real k,

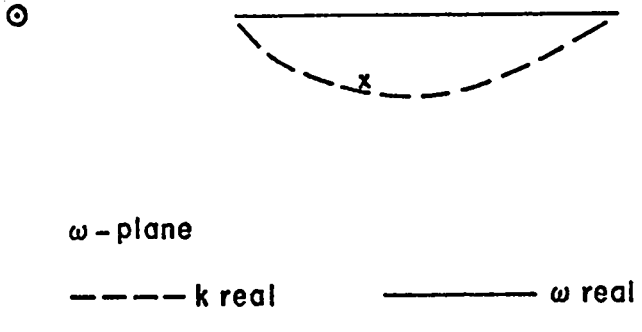

Fig. 5. The complex ω plane. 0 = origin,--- line of real k, x = a possible value of z, solid line: branch cut along line of real ω.

hence complex ω with negative imaginary part, see Eq. (3.2) and Fig. 5. With ω_k thus defined, we have a unitarity relation[37] along this line. If we take[38] the free-space unitarity relation of Chew and Low [their Eq. (34)],[39] then

$$\lim_{z\to\omega_k+i\varepsilon} h_\alpha(z) =$$

$$= e^{i\delta_\alpha(k)} \sin \delta_\alpha(k)\mu^2/k^3v^2(k) . \qquad (3.7)$$

The point z is just on the positive imaginary side of the branch cut, so z still has a negative imaginary part. The amplitude itself is, of course, at present still unknown, only the form of the amplitude is dictated by unitarity.

By considering the singularities of $h_\alpha(\omega)$, Chew and Low derive their Eq. (40) for $h_\alpha(z)$. They can greatly simplify this by introducing a new function g_α, setting

$$h_\alpha(z) = \lambda_\alpha/[zg_\alpha(z)] , \qquad (3.8)$$

where λ_α is related to the unrationalized coupling constant between pion and nucleon, $f^2 \approx 0.08$. In particular,

$$\lambda_3 = \frac{4}{3} f^2 , \qquad (3.9)$$

whereas λ_1 and λ_2 are negative and therefore lead to repulsive interactions in states 1 and 2. For $g_\alpha(z)$, Chew and Low then obtain the dispersion relation

$$g_\alpha(z) = 1 - \frac{z}{\pi}\int_\mu^\infty dx' \left[\frac{F_\alpha(x')}{x'-z} + \frac{G_\alpha(x')}{x'+z}\right] . \qquad (3.10)$$

We obtain precisely the same relation with the only difference being that the integral goes along a complex path in the x'-plane, viz., along the dashed line of Fig. 5 which corresponds to real values of k.

To determine F_α, we use the unitarity condition in Eq. (3.7) [see also Appendix C, Eq. (C.18) and Eq. (C.20)] and the definition of g_α in Eq. (3.8) which give

$$F_\alpha(\omega_k) = -\frac{1}{2i\omega_k}\left[\lim_{z\to\omega_k+i\eta} g_\alpha(z) - \lim_{z\to\omega_k-i\eta} g_\alpha(z)\right]$$

$$= \frac{\lambda_\alpha}{\omega_k^2} k^3 v^2(k)\mu^{-2} \quad . \tag{3.11}$$

The function G_α is determined in turn from the crossing relations. Once we know the path in the complex ω-plane and the (real) value k at each point on the path, we may evaluate Eq. (3.10) to find $g_\alpha(z)$ for <u>any</u> complex value of z.

To find ω_k we must find the poles of Eq. (2.1) for values of ω just above the cut in Fig. 5, which means evaluating $g_\alpha(\omega)$ for these same values of ω. But then the integration in Eq. (3.10) encounters the pole, so we may write g_α as the sum of a pole term Λ_1 and a remainder Λ_2, which contains the principle value integration (denoted by P),

$$\Lambda_1(\omega_k) = -i\lambda_\alpha k^3 v^2(k)/\omega_k\mu^2 \tag{3.12}$$

$$\Lambda_2(\omega_k) = 1 - \frac{\omega_k}{\pi} P\int dx' \left[\frac{\lambda_\alpha k'^3 v^2(k')\mu^{-2}}{x'^2(x'-\omega_k)} + \right.$$

$$\left. + \frac{G_\alpha(x')}{x'+\omega_k}\right] \quad . \tag{3.13}$$

The integral [Eq. (3.13)] clearly diverges linearly for large x' until the form factor v(k') cuts it off. Hence the main contribution comes from large x', and therefore the value of the integral does not depend strongly on ω_k. This is Chew and Low's effective range approximation. The integral has the same form as in Ref. 16; the only difference is that the path of integration is now complex. We may thus write

$$\Lambda_2(\omega_k) \approx 1 - \omega_k(r_1 + ir_2) \quad . \tag{3.14}$$

But the imaginary part of x' is only important for small x' (see below), hence the imaginary part of the integral will be small compared with the real;

$$r_1 \gg r_2 \quad . \tag{3.14a}$$

Further, r_1 will be almost equal to the "effective range" of Chew and Low,

$$r_1 \approx r \quad . \tag{3.14b}$$

As $\rho \to 0$, of course, $r_2 \to 0$, and $r_1 \to r$. The Chew-Low effective range r is directly related to the resonance energy

$$r = 1/\omega_r \quad , \tag{3.14c}$$

for which experiments give the result[41]

$$\omega_r = 2.39\ \mu = 334\ \text{MeV} \quad . \tag{3.14d}$$

To see that the imaginary part of x' is unimportant, start with large k. The Rayleigh scattering formula [or our main result, Eq. (3.26)] shows that for large k

$$|f(k,0)| < (2\ell + 1)/k \quad . \tag{3.15}$$

For the $p_{3/2}$ state $2\ell + 1$ is replaced by $j + \frac{1}{2} = 2$. Using Eq. (2.11),

$$|\Pi(k)| < 8\pi\rho/k \quad . \tag{3.16}$$

Therefore, using Eq. (2.2)

$$|\omega_k^2 - \omega_{k0}^2| = |\Pi(k)| < 8\pi\rho/k \quad . \tag{3.17}$$

The same limit holds for Im ω_k^2. Therefore

$$\frac{\text{Im}(\omega_k)}{\omega_k} < 4\pi\rho/k^3 \tag{3.18}$$

making use of the fact that $\omega_k \approx k$ for large k. The condition for

$$\frac{\text{Im}(\omega_k)}{\omega_k} \ll 1 \tag{3.19}$$

is that

$$k_c^3 \equiv 4\pi\rho \ll k^3 \quad . \tag{3.20}$$

Using nuclear matter density $\rho_o = 0.16\ \text{fm}^{-3} = 0.47\mu^3$;

$$k_c = 1.81\,\mu \quad ,$$

which is less than the momentum of a free pion at resonance $k_r = 2.2\,\mu$. Equation (3.20) shows that $\mathrm{Im}\omega_k < \omega_k$ already when k is only moderately larger than k_c, hence over most of the important range of integration in Eq. (3.13).

Concerning the quantitative side, Dover and Lemmer[15] have shown that the cut-off v(k) must occur at k = 12 to 14 μ if we want to reproduce[40] the position of the resonance without using the crossing terms. On the other hand, r_2 comes from a range of order $k_c - \mu \approx \mu$; so we might expect

$$r_2/r_1 < 0.1 \quad . \tag{3.21}$$

Thus, as a first approximation we may set $r_2 = 0$ and use Eqs. (3.14b) and (3.14c) for r_1. Equations (3.10), (3.12), and (3.13) then give

$$g_\alpha(\omega_k) = 1 - \omega_k/\omega_r - i\lambda_\alpha k^3/\omega_k\mu^2 \quad . \tag{3.22}$$

Finally we use g to calculate the scattered amplitude. Using Eqs. (3.4), (3.5), and (3.8) with q' = q (note that Eq. (3.4) is symmetric in q, q' as it should be) we find

$$t_{q'q}(z) = -v^2(q)\,\frac{2\pi}{\omega_q}\,\frac{\lambda_3}{z g_3(z)}\,\frac{2q^2}{\mu^2}\left(\delta_{q'q} - 1/3\;\tau_q\tau_{q'}^{\dagger}\right) \quad . \tag{3.23}$$

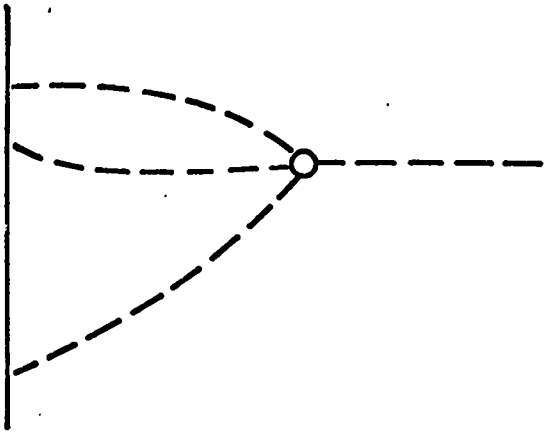

Fig. 6. The Lowest mass perturbation of the pion-nucleon vertex, according to Hamilton and Braathen, Ref. 40.

The isospin factor is (3.24a)

$$C_{q'q} = \delta_{q'q} - 1/3\;\tau_q\tau_{q'}^{\dagger} = \begin{cases} 1 & \text{for } P\pi^+ \text{ and } N\pi^- \\ 1/3 & \text{for } P\pi^- \text{ or } N\pi^+ \\ 2/3 & \text{for } P\pi^0 \text{ or } N\pi^0 \\ 2/3 & \text{for an average nucleon, and an average pion of any charge.} \end{cases}$$

For g_3 given by Eq. (3.22), we have

$$t_{qq}(\omega_k) = -v^2(q)\,\frac{2\pi}{\omega_q}\,\frac{\lambda_3 C_{qq}}{\omega_k(1 - \omega_k/\omega_r) - i\lambda_3 k^3} \times \times \frac{2q^2}{\mu^2} \quad . \tag{3.24b}$$

The forward scattering amplitude can be shown to be

$$f(k,k) = -(\omega_k/2\pi)\;t_{kk}(\omega_k + i\varepsilon) \tag{3.25}$$

where we have set q = k, the wave number of a pion of energy ω_k moving in the +z direction. (Note that the substitution q = -k leads to the same result, showing that the theory gives correct results for pions moving to the left and to the right). Therefore, we have

$$f(k,k) = C_{kk}\,\frac{2\lambda_3}{\mu^2}\,\frac{k^2 v^2(k)}{\omega_k g_3(\omega_k)} \quad . \tag{3.26}$$

Using g_3 obtained from Eq. (3.22) we may now locate the poles of Eq. (2.1) to find ω_k in the first approximation. We assume here that there is one and only one important or dominant solution ω_k; our calculations show that there is only one continuous solution for ω = real which has a physically reasonable form. The value ω_k may be used in the integrand of Eq. (3.10) to find an improved g_3 which may be used in turn in Eqs. (3.26), (2.11), and (2.1) to find an improved ω_k. The procedure presumably converges fairly fast to the true ω_k.

As we have said, the solution of Eq. (2.2) relevant to nuclear matter is the solution k_ω for ω = real, which is the analytic continuation of the solution found above for k = real. To find this so-

lution we need $g_\alpha(\omega)$ for ω = real, which is obtained from Eq. (3.10) once we have found the solution ω_k to the above problem. We may then use this $g_3(\omega)$ in Eq. (3.25), set $k = k_\omega$ and use Eqs. (2.11) and (2.2) to get the desired solution.

We may, however, go one step further. $F_\alpha(\omega)$, as given in Eq. (3.11), is clearly an analytic function of k. Therefore, we may change the path of integration in Eq. (3.10) from the line of real k to the line of real ω, i.e., from the dashed line in Fig. 5 to the solid line. We merely need to replace all terms in Eq. (3.11) by their analytic equivalents, i.e., ω becomes real and k_ω complex. We can now determine $g_\alpha(\omega)$ from Eq. (3.10). There is again the contribution from the pole, which is

$$- i\omega F_\alpha(\omega), \tag{3.27}$$

and that from the principal value. The latter is now easier to handle because the integral is along the real axis x'. Further, since our F_α is the analytic continuation of that quantity for real k, the same should be true for the integral. Using again the Chew-Low effective range approximation, we get

$$\frac{1}{\pi} P \int_\mu^\infty dx' \left[\frac{F_\alpha(x')}{x' - z} + \frac{G_\alpha(x')}{x' + z}\right] = r_1 + ir_2 \ . \tag{3.28}$$

The real part is now even more similar to Chew and Low so $r_1 \approx r = 1/\omega_r$ [Eqs. (3.14b, 3.14c)], but we retain r_1 because the small difference from r may be interesting for the shift of the resonance energy. Inserting Eqs. (3.27 - 28) into Eq. (3.10),

$$g_\alpha(\omega) = 1 - \omega r_1 - \frac{i}{\omega}\left[\lambda_\alpha k_\omega^3 v^2(k_\omega)\mu^{-2} + \omega^2 r_2\right] \ . \tag{3.29}$$

Equation (3.29) has the Chew-Low form. In the expression with -i, which is essentially imaginary, the main term is the first, which is proportional to k_ω^3. This is just the Chew-Low term, but with k in the medium substituted. This agrees with the theories of Barshay and collaborators,[18,19] and disagrees with all other theories. The last term r_2 is a small correction. The bracket gives the "width of the resonance". We see that on the low-energy side this is much greater than for free π-nucleon scattering. This is in contrast to some conclusions of some earlier papers,[15,17] but is in agreement with experiments.

The real part of g goes to zero at resonance (by definition, see Sec. VIII). There is a real contribution to g from the term with i, namely,

$$\frac{\lambda_\alpha \mu^{-2}}{\omega} \operatorname{Im} k_\omega^3 \ , \tag{3.30}$$

which is, in general, positive. This pushes the zero of Eq. (3.29) to higher energy. The "range" r is of course also slightly changed (probably increased) from the Chew-Low value. But the definition of "resonance" in nuclear matter is rather complicated (Sec. VIII).

We next use g to calculate the scattered amplitude. Again use Eq. (3.25) for this, and find

$$f_3(k,k) = C_{kk} \frac{2\lambda_3}{\mu^2} \frac{k_\omega^2 v^2(k_\omega)}{\omega(1 - \omega r_1) - i[\ ]} \ , \tag{3.31}$$

where [ ] is the square bracket in Eq. (3.29). To Eq. (3.31) has to be added

1. the scattered amplitude for the 11, 13, and 31 states,
2. the scattered amplitude for s-states, and
3. the absorption of pions by the two-nucleon process of Eq. (1.11).

Amplitudes 1 and 2 can be taken from the free pion-nucleon scattering. Amplitude 3 can be taken from the theory of the Ericsons.[5] The sum then must be integrated over ρ to get Π [Eq. (2.11)]. We then find k_ω from Eq. (2.2), and insert this back into Eq. (3.31). The process must be repeated until self-consistency is achieved between the k_ω used in Eq. (3.31) and that computed from Eqs. (3.31), (2.11), and (2.2).

Equation (3.31), with the bracket replaced by its first term [see Eq. (3.29)] and r_1 by $1/\omega_r$, is identical with Barshay et al.[19] [their Eq. (5)].

IV. PAULI PRINCIPLE

In treating pion scattering, we have so far disregarded the behavior of the nucleon which does the scattering. Actually, this nucleon (a) receives energy from the pion, and (b) is subject to the Pauli principle.

The Pauli principle, in nuclear matter, means that the nucleon can go only (a) into states of momentum[28]

$$p > p_F \quad , \tag{4.1}$$

where p_F is the Fermi momentum, or (b) into the state from which it came. Scattering into a state $p > p_F$ is "quasi-elastic" and always involves energy loss of the pion; elastic scattering means to return to the initial state.

Figure 7 shows a typical Feynman diagram of the type considered by Chew and Low.[16] In the initial state, nucleon and pion have the momenta $\vec{p}_i$ and $\vec{k}_i$, respectively, so that the total momentum is

$$\vec{P} = \vec{p}_i + \vec{k}_i \quad . \tag{4.2}$$

this $\vec{P}$ is conserved. In the intermediate states a, ...e, at least one intermediate pion is present. Since the Chew-Low theory (Sec. III) states that the intermediate pion states of large momentum are most important, the intermediate nucleon momenta $P_a, \cdots P_e$ will nearly all satisfy the condition of Eq. (4.1). There is therefore little influence of the Pauli principle in intermediate states.[42] Eisenberg and Weber[17] have treated this problem in their approximation; we shall neglect this correction in this report.

The final state of the pion has a momentum

$$k_f \leq k_i \quad , \tag{4.3}$$

hence there is a considerable probability that the final nucleon momentum

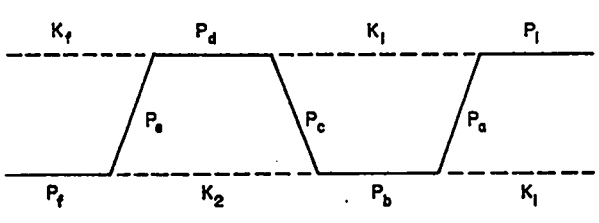

Fig. 7. The Chew-Low diagram in nuclear matter. Solid line nucleon, dashed line pion.

$$\vec{p}_f = \vec{P} - \vec{k}_f \tag{4.4}$$

does not satisfy Eq. (4.1). Such final states are forbidden. Now, by the optical theorem, the imaginary part of the elastic, forward scattering amplitude is proportional to the cross section of quasi-elastic scattering, disregarding true absorption (See Sec. VI). This cross section being diminished, Im f(k,k) will also be, and so will be the imaginary part of g [Eq. (3.11)]. We therefore have to re-examine the unitarity relation.

This is conveniently done by the Lippman-Schwinger equation.[43] We define the K matrix which is given by the same diagrams as T, but using the denominators $E - E_n$ without $i\varepsilon$, so that K is given by the principal value of the dispersion integrals. In this section, we shall assume k to be unchanged in quasi-elastic scattering which amounts to the assumption of infinite mass of the nucleon. This assumption will be corrected in Sec. V. Further, just as in the main part of Sec. III, we assume k to be real. Then in the medium the matrices T and K for different directions $\vec{k}_i$, $\vec{k}_f$ of k, are related by (see App. C)

$$\langle \vec{k}_f | T | \vec{k}_i \rangle = \langle \vec{k}_f | K | \vec{k}_i \rangle - (2\ell+1) i (k\omega/8\pi^2) \times \int \langle \vec{k}_f | K | \vec{k} \rangle \, Q(\vec{k}) \, d\Omega \langle \vec{k} | T | \vec{k}_i \rangle \tag{4.5}$$

for scattering with orbital momentum ℓ. The integral goes over all directions of $\vec{k}$. We shall assume $\ell = 1$, and set[44]

$$\langle \vec{k}_f | K | \vec{k} \rangle = -2\pi \, K_o(\omega) \cos\theta \quad , \tag{4.6}$$

where θ is the scattering angle, i.e., in this case the angle between $\vec{k}$ and $\vec{k}_f$, and K_o is a function of ω only, hence constant in Eq. (4.5). K and T are of dimension (length)^2 as is also clear from Eqs. (3.4) and (3.7). Later on, in Eq. (4.26), we shall use for K the correct $p_{3/2}$ type angular distribution. The factor Q(k) in Eq. (4.5) takes care of the Pauli principle,

$$Q = 1 \text{ if } p = |\vec{P} - \vec{k}| > p_F$$
$$Q = 0 \text{ if } p < p_F \quad . \tag{4.7}$$

In the absence of the Pauli principle, T will have the angular dependence

$$T = -2\pi\, T_0 \cos\theta \quad . \tag{4.8}$$

Inserting this into Eq. (4.5) clearly solves the equation for all θ, and

$$T_0 = K_0 + ik\omega\, K_0\, T_0 = \frac{K_0}{1 - ik\omega\, K_0} \quad . \tag{4.9}$$

If $K_0 = (k\,\omega)^{-1} \tan\delta$, then

$$T_0 = (k\,\omega)^{-1} \sin\delta\, e^{i\delta} \tag{4.10}$$

as it should be.

In the presence of the Pauli principle, T is clearly no longer proportional to $\cos\theta$. It might be expanded into a series of spherical harmonics in θ and ϕ, but this would be clumsy. We shall instead solve Eq. (4.5) explicitly as a function of angle.

It is clearly reasonable to introduce a polar coordinate system with $\vec{P}$ [Eq. (4.2)] as the axis. Let the polar angles be χ and ψ, with $\psi = 0$ in the plane of $\vec{P}$ and $\vec{k}_i$. Then the condition [Eq. (4.7)] for the intermediate state $\vec{k}$ to be allowed is

$$\cos\chi < \frac{P^2 + k^2 - p_F^{\,2}}{2Pk} \equiv \cos\chi_2 \equiv \lambda \quad . \tag{4.11}$$

Now according to Eq. (2.11), the only scattered amplitudes needed are those for which

$$p_i = |\vec{P} - \vec{k}_i| = p_F \quad . \tag{4.11a}$$

Further, because we have assumed [see above Eq. (4.5)] that k remains unchanged in the scattering, we have $k = k_i$. But then it is clear that

$$\chi_2 = \chi_i \quad , \tag{4.11b}$$

which is the angle between $\vec{P}$ and $\vec{k}_i$. This angle always exists, so we always have

$$-1 \leqslant \lambda = \cos\chi_2 < 1 \quad . \tag{4.12}$$

Note that the upper limit in Eq. (4.12) would also hold if $p_i < p_F$ because

$$P^2 + k^2 - 2Pk = (P - k_i)^2 \leqslant (\vec{P} - \vec{k}_i)^2 = p_i^{\,2} < p_F^{\,2} \quad ; \tag{4.13}$$

but the lower limit condition, $\cos\chi_2 \geq -1$, would not always hold. Indeed, Eq. (4.11) gives $\lambda < -1$ if

$$P + k < p_F \quad . \tag{4.13a}$$

On the other hand, if $k < p_i$, the smallest value of P permitted is

$$P_{min} = p_i - k \quad , \tag{4.13b}$$

which clearly satisfies Eq. (4.13a) if $p_i < p_F$. There is therefore a range of P for which $\lambda < -1$; in this case, the Pauli principle excludes <u>all</u> quasi-elastic scattering. One then has to use Eq. (4.11) for $P > p_F - k$, and $\lambda = -1$ for $P < p_F - k$; in other words, the Pauli limit is not an analytic function of P. With $p_i = p_F$, Eq. (4.11) is <u>always</u> valid, and the Pauli limit is analytic.

Using Eq. (4.11), the integral in Eq. (4.5) becomes simply

$$\int_{-1}^{\lambda} d(\cos\chi) \int_0^{2\pi} d\psi\ \langle\vec{k}_f|K|\vec{k}\rangle \langle\vec{k}|T|\vec{k}_i\rangle \quad . \tag{4.14}$$

The K matrix element here is [see Eq. (4.6)]

$$\langle\vec{k}_f|K|\vec{k}\rangle = -2\pi\, K_0 \left[\cos\chi_f \cos\chi + \sin\chi_f \sin\chi \cos(\psi - \psi_f)\right] . \tag{4.15}$$

It is easy to see that T must have the form

$$\langle\vec{k}|T|\vec{k}_i\rangle = T_1(\chi) + T_2(\chi)\cos\psi \tag{4.16}$$

and must not contain any terms proportional to $\cos n\psi$ with $n > 1$, because only in this way can the integral over ψ in Eq. (4.14) give a nonvanishing result. Carrying out this integral, we obtain sepa-

rate equations for T_1 and T_2.

$$T_1(\chi_f) = -2\pi K_0 \cos\chi_i \cos\chi_f + (3i/2)\, k\omega K_0 \cos\chi_f \int_{-1}^{\lambda} d(\cos\chi)\, T_1(\chi) \cos\chi . \tag{4.17}$$

$$T_2(\chi_f) = -2\pi K_0 \sin\chi_i \sin\chi_f + (3i/4)\, k\omega K_0 \sin\chi_f \int_{-1}^{\lambda} d(\cos\chi)\, T_2(\chi) \sin\chi . \tag{4.18}$$

Evidently, the integrals do not depend on χ_f, so we may set

$$T_1(\chi_f) = -2\pi T_C \cos\chi_i \cos\chi_f$$

and

$$T_2(\chi_f) = -2\pi T_S \sin\chi_i \sin\chi_f . \tag{4.19}$$

Inserting these into the integrals, we get the <u>algebraic</u> equation

$$T_C = K_0 + i K_0 T_C k\omega \tfrac{3}{2} \int_{-1}^{\lambda} \cos^2\chi\, d(\cos\chi) = K_0 + \tfrac{1}{2} i K_0 T_C k\omega (1 + \lambda^3) \tag{4.20}$$

or,

$$T_C = \frac{K_0}{1 - \frac{1}{2} i K_0 k\omega (1 + \lambda^3)} , \tag{4.21}$$

where λ is given in Eq. (4.11). If the lower limit of cos χ were λ', we would have in the denominator

$$\lambda^3 - \lambda'^3 \text{ instead of } 1 + \lambda^3 . \tag{4.22}$$

Similar to Eq. (4.21),

$$T_S = \frac{K_0}{1 - \frac{1}{4} i K_0 k\omega (1 + \lambda)(2 + \lambda - \lambda^2)} . \tag{4.23}$$

Thus we have solved the scattering problem with the Pauli principle.

In the limit when there is no Pauli restriction, $\lambda = +1$, and the denominators of Eqs. (4.21) and (4.23) both reduce to

$$1 - iK_0 k\omega , \tag{4.24}$$

as they should.

The forward scattering amplitude, $\chi_f = \chi_i$, $\psi = 0$, is now

$$\langle \vec{k}_i | T | \vec{k}_i \rangle = - \frac{2\pi K_0 \cos^2\chi_i}{1 - \frac{1}{2} i K_0 k\omega (1 + \lambda^3)} - \frac{2\pi K_0 \sin^2\chi_i}{1 - \frac{1}{4} i k\omega K_0 (1 + \lambda)(2 + \lambda - \lambda^2)} . \tag{4.25}$$

The actual case of the $p_{3/2}$ state is treated in Appendix A, and gives very similar results. The scattering matrix is given in Eq. (A.16) and may be written

$$\langle \vec{k}_i\, m | T | \vec{k}_i\, m \rangle = -\pi K_0 \left[\frac{3 \sin^2\chi_i}{1 - \frac{1}{4} i k\omega K_0 (1+\lambda)(2+\lambda-\lambda^2)} + \frac{3\cos^2\chi_i + 1}{1 - \frac{1}{4} i k\omega K_0 (1+\lambda)(2-\lambda+\lambda^2)} \right] . \tag{4.26}$$

Here m is the component of the spin in the direction of $\vec{P}$. Equation (4.26) gives the same result for m = + 1/2 and - 1/2. Also there is no "spin flip" from + 1/2 to - 1/2. Because of Eq. (4.11b),

$$\cos\chi_i = \lambda , \quad \sin^2\chi_i = 1 - \lambda^2 . \tag{4.26a}$$

We have used real values of k in the derivation of Eqs. (4.25) and (4.26), just as in Eq. (3.11). In our case, there is added reason for this; the nucleon momenta are real, hence the momentum conservation equations, like Eq. (4.4), can be fulfilled only if k is also real. After having established Eq. (4.26) for real k, we continue the same way as in Sec. III. In particular, we use the analytic continuation to real values of ω. For this it is important that $\lambda = \cos\chi_2$ is an analytic function of k and P, and T is an analytic function of λ. But, as explained below Eq. (4.13), λ will not be an analytic function of k and P if $p_i < p_F$; therefore, it is important for the success of the theory that we proved in

Eq. (2.11) that only matrix elements for $p_i = p_F$ are required.

T in Eq. (4.26) is a function of k, ω, and P (because λ occurs). K_0 is a function of k and ω only, and is obtained from Sec. III. T must then be integrated over all values of P [see Eq. (4.2)] corresponding to all directions of $\vec{p}_i$ relative to $\vec{k}$. It will be remembered that Eq. (2.9) requires such an integration. We have

$$f_{Av} = \frac{1}{4\pi}\int d\Omega_p\, f(P) = \frac{1}{4k\, p_F}\int_{|p_F-k|}^{p_F+k} PdPf(P). \quad (4.27)$$

The lower limit in Eq. (4.27) holds for real k. When we extend the theory to real ω and complex k, it is simplest to assume that

$$\text{Im } P = \text{Im } k \quad (4.28)$$

and to take the lower limit in Eq. (4.27) to be $k - p_F$.

V. ENERGY LOSS

In this section, we treat the effects of the finite mass of the nucleon. Once we take this fact into account, a pion scattered by a single nucleon transfers energy to the latter; the scattering is called "quasi-elastic".

We must first discuss frames of reference. Three frames should be considered: (a) the frame in which our relation between ω and k holds, (b) that in which the Chew-Low dispersion theory works, (c) the frame in which experimental results on the position of the resonance for scattering by a nucleon and by a complex nucleus are to be compared.

(a) The relation between k and ω holds for the nucleus as a whole. Therefore it holds in the center-of-mass system of the entire nucleus and the pion. As the nuclear mass increases and we approach nuclear matter, this frame becomes identical with the laboratory frame. Only in this frame can we speak of a definite relation between ω and k of the pion. If we wanted such a relation in the center-of-mass system of the pion and an individual nucleon, it would be different for each nucleon momentum $\vec{p}$, hence not definite.

(b) The Chew-Low equation holds in the center-of-mass (cm) system of the pion and the scattering nucleon. The Chew-Low theory is concerned only with these two bodies. The rest of the nucleus may influence the relation between energy and momentum, both of the nucleon and the pion, but it will not enter the scattering process. Because frames (a) and (b) are different, we shall have to make a Lorentz transformation connecting them. (Because we use the cm system, the relation between energy and momentum of the pion, even at high energy, is not strictly $\omega = (k^2 + \mu^2)^{\frac{1}{2}}$, as we assumed in Sec. III, but it is still the same in the free-nucleon and the nuclear-matter case.)

(c) In comparing experimental results, it is essential to use the same reference frame for scattering by the complex nucleus and by the nucleon. Since, as was discussed in (a), the laboratory frame should be used for the nucleus case, the same frame must also be used for the nucleon.[45] If, e.g., the theory were to predict that the resonance energy is not shifted, then it should be observed at the same laboratory energy in complex nuclei as for a nucleon, i.e., at a pion kinetic energy of 194 MeV. It is for this reason that we have used ω_r in the laboratory in Eq. (3.14d).

To calculate the energy loss of the pion, we must know the relation between energy and momentum for the nucleon. As we discussed in Eq. (2.5), the nucleon energy depends on the momentum p not only because of the kinetic, but also because of the potential energy U(p). This dependence is most conveniently expressed in terms of an effective mass M^*, a concept introduced by Brueckner; then

$$\varepsilon(\vec{p}) = p^2/2M^* + U_0 \quad . \quad (5.1)$$

This expression was found to be a very good approximation in many nuclear matter calculations. Siemens[46] has also determined the dependence of M^* on density, namely,

$$M/M^* \equiv C_m = 1 + \alpha\, p_F{}^2 \quad . \quad (5.2)$$

If p_F is measured in fm^{-1}, Siemens finds $\alpha = 0.27$, but we believe 0.24 would be better[46] [see Eq. (5.2b)]. For normal nuclear density, $p_F = 1.33$,

hence $C_m = 1.43$. The potential energy for zero momentum U_0 is found from the binding energy B, namely

$$U_0 + p_F^2/2M^* = -B(p_F) \quad . \tag{5.3}$$

The binding $B(p_F)$, for $p_F \neq 1.33\ \mathrm{fm}^{-1}$, may be taken from nuclear matter calculations,[29] but we shall not actually need U_0 in our theory. For normal nuclear density, B = 16 MeV,

$$p_F^2/2M^* = 53\ \mathrm{MeV}, \text{ hence } U_0 = -69\ \mathrm{MeV} \ . \tag{5.3a}$$

Using Eq. (5.1), and working in the laboratory system (or better the system in which the whole nucleus is initially at rest), we have for the total energy of nucleon and pion

$$W = E_{ni} + \omega_i = M + p_i^2/2M^* + U_0 + \omega_i \tag{5.4}$$

where the subscript i denotes the initial state, as before, and the nucleon mass energy $M = Mc^2$ has been included. The energy W is, of course, conserved in the quasi-elastic scattering. Because $p_i = p_F$, we have from Eq. (5.3) that

$$p_i^2/2M^* + U_0 = -B(p_F) \quad , \tag{5.4a}$$

which is only about 16 MeV, and hence negligible compared with M in Eq. (5.4).

For any meson state $\vec{k}$ which can be reached by quasi-elastic scattering, we have the nucleon momentum

$$\vec{p} = \vec{P} - \vec{k} \tag{4.4a}$$

[see Eqs. (4.4) and (4.2)]. We assume that $\vec{k}$ is in the direction χ, ψ with respect to $\vec{P}$. The nucleon energy is then

$$\begin{aligned} E_n &= M + (\vec{P} - \vec{k})^2/2M^* + U_0 \\ &= M + U_0 + (2M^*)^{-1} (P^2 + k^2 - 2Pk \cos\chi) \ . \end{aligned} \tag{5.5}$$

Energy conservation then requires that the meson energy $\omega(k)$ satisfy the equation

$$E_n + \omega(k) = W \ . \tag{5.6}$$

Inserting Eqs. (5.4) and (5.5), and cancelling $M + U_0$, we get

$$\begin{aligned} &\omega(k) + (2M^*)^{-1} (P^2 + k^2 - 2Pk \cos\chi) \\ &\quad = \omega_i + p_i^2/2M^* \ . \end{aligned} \tag{5.7}$$

In our laboratory (or nucleus) system, we have a definite relation between ω and k, just as in Sec. III. Therefore, given ω_i, p_i and P, Eq. (5.7) gives a definite relation[47] between χ and ω. For any final energy $\omega(k)$ of the pion, there is a definite angle χ between $\vec{P}$ and $\vec{k}$. In Appendix B, we find that, in general, $\cos\chi$ can go from the maximum of $\cos\chi_2$ [Eq. (4.11)] to -1. The latter value corresponds to the minimum possible value of ω, and we shall, in general, have

$$\mu < \omega_{min} < \omega_i \ . \tag{5.8}$$

It can easily be seen that, because $p_i = p_F$, Eq. (5.7) is fulfilled for $k = k_i$, $\omega(k) = \omega_i$, and $\chi = \chi_2$ [Eq. (4.11)]. There is a continuous range of permissible values of ω from ω_{min} to ω_i.

So far, we have worked in the laboratory system because only in that system is there a definite relation $\omega(k)$. However, to apply the Chew-Low theory, we should now go into the center-of-mass system of pion and nucleon. We shall denote quantities in the cm system by primes. We consider $\vec{P}$ and ω_i as fixed. The cm system moves relative to the laboratory system with the velocity

$$\vec{\beta} = \frac{\vec{P}}{W} \approx \frac{\vec{P}}{M + \omega_i} \quad , \tag{5.9}$$

where W is the total laboratory energy, and where we have neglected the binding energy [Eq. (5.4a)] in the expression for W [Eq. (5.4)]. Setting[48] $\gamma = (1 - \beta^2)^{-\frac{1}{2}} = 1$, the Lorentz transformation gives for the pion momentum in the cm system after scattering

$$\vec{k}' = \vec{k} - \omega(k)\,\vec{\beta} \quad . \tag{5.10}$$

We write its components as

$$k_\ell' = k' \cos\chi' = k\cos\chi - \omega(k)\,\beta \tag{5.11}$$

and

$$k_t' = k' \sin\chi' = k \sin\chi \tag{5.12}$$

(ℓ = longitudinal, t = transverse to $\vec{\beta}$, χ' = angle between $\vec{P}$ and $\vec{k}'$). From this we get

$$k'^2 = k^2 + \omega^2\,\beta^2 - 2k\omega\beta\cos\chi \quad . \tag{5.13}$$

By Eq. (5.7), cos χ is a function of ω alone (P and ω_i being fixed), therefore k' and cos χ' are likewise functions of the single variable ω. (The term $\omega^2\beta^2$ in Eq. (5.13), being of second order in β, can generally be neglected.)

The cm pion energy is, setting again $\gamma = 1$

$$\omega' = \omega - \vec{k}\cdot\vec{\beta} \approx \omega - k\beta\cos\chi \quad . \tag{5.14}$$

In the total cm energy, we set $\gamma = 1 + \frac{1}{2}\beta^2$ and have[49]

$$W' = W\gamma^{-1} \approx W - \frac{p^2}{2(M+\omega_i)} \quad . \tag{5.15}$$

W' is, of course, conserved in scattering. But ω' is, in general, <u>not</u> conserved because the pion energy $\omega(k)$ is a complicated function of the momentum rather than a simple quadratic function. Likewise, the nucleon energy in the cm by itself is not conserved. But conservation of W' is enough for us, and we identify this quantity with the cm energy in the scattering of a pion by a <u>free</u> nucleon which occurs in the Chew-Low theory.

The use of the cm system is necessary because it is in this system that the π-nucleon scattering has a simple angular dependence, and in this system the Lippman-Schwinger equation holds. We get precisely Eq. (4.5), only with primes, $\vec{k}'_i$, $\vec{k}'$, $\vec{k}'_f$. The Pauli principle limit is still Eq. (4.11) in the lab system. Just in the limit, when $\chi = \chi_2$, Eq. (5.7) shows that $k = k_i$, and then Eq. (4.11b) also still holds.[50]

Therefore Eqs. (4.5) and (4.14) together become for $\ell = 1$

$$\langle\vec{k}_f'|T|\vec{k}_i'\rangle = \langle\vec{k}_f'|K|\vec{k}_i'\rangle + (3i/8\pi^2) \times \int_1^{\cos\chi_i'} \omega'\,k'\,d(\cos\chi')\int d\psi' \;\langle\vec{k}_f'|K|\vec{k}'\rangle\langle\vec{k}'|T|\vec{k}_i\rangle \quad , \tag{5.16}$$

where $\cos\chi_i'$ may be obtained from Eqs. (5.11) and (5.13), and $\psi' = \psi$.

Now assuming a pure p-state interaction and disregarding spin factors, we may set

$$\langle\vec{k}_f'|K|\vec{k}'\rangle = -2\pi K_1(W')\,\vec{k}_f'\cdot\vec{k}'/\mu^2 \approx -2\pi K_1(W')\,k_f'\,k'\,\mu^{-2}\left[\cos\chi_f'\cos\chi' + \sin\chi_f'\sin\chi'\cos(\psi_f - \psi)\right] \quad , \tag{5.17}$$

where $K_1(W')$ is obtainable from the Chew-Low equation. Equation (5.17) differs from Eq. (4.15) by the factor $k_f' k'/\mu^2$, which, in Eq. (5.17), is explicitly exhibited while in Eq. (4.15) it is included in the definition of K. We write T in the same way as Eq. (4.16) but modify the solution [Eq. (4.19)] to read

$$T_1(\chi_f) = -2\pi T_C\,k_i'\cos\chi_i'\,k_f'\cos\chi_f'\,\mu^{-2} \tag{5.18}$$

and similarly for T_2. The same argument as in Eqs. (4.20) and (4.21) leads to

$$K_1(W')/T_C(W') = 1 - \tfrac{3}{2}\,i\,K_1(W') \times \int_{-1}^{\cos\chi_i'} \omega'\,k'\,d(\cos\chi')\,k'^2\cos^2\chi'\,\mu^{-2} \quad . \tag{5.19}$$

Here we may express k' cos χ' by Eq. (5.11), and further write

$$k'\,d(\cos\chi') = d(k'\cos\chi') - (k'\cos\chi')\,d(k'^2)/2k'^2 \quad . \tag{5.20}$$

This is convenient because we have simple expres-

sions[51] for k' cos χ' and k'², [Eqs. (5.11) and (5.13)].

It is clear that the integral in Eq. (5.19) is smaller than Eq. (4.20), simply because generally

$$\omega' < \omega_i \ , \ k' < k_i \quad . \tag{5.20a}$$

This is physically reasonable. The integral in Eqs. (4.20) and (5.19) represents the "damping" of the elastic scattering by the quasi-elastic one. Since the latter involves a decrease of momentum, and since the "emission" of a p-state pion is proportional to k^3, the reduced momentum means reduced damping. The Pauli principle (Sec. IV) already reduced the damping compared with free-nucleon scattering (Sec. III). Some damping will always remain.

The expression for T_S [Eq. (4.19)], is similar to Eq. (5.19), except that $k'^2 \cos^2 \chi'$ is replaced by $(k' \sin \chi')^2 = k^2 \sin^2 \chi$. In the model we shall discuss below, we shall consider instead of T_C and T_S an average, namely, we replace $3(k' \cos \chi')^2$ in Eq. (5.19) by k'^2. This will simplify the calculation.

We shall now make an estimate of the energy transferred to the nucleon, assuming that the magnitude of the pion momentum does not change (we shall soon see that this is not an unrealistic assumption). The greatest energy transfer clearly occurs when the pion is scattered backward; it is

$$\Delta E = \frac{(2\vec{k} + \vec{p}_i)^2 - p_i^2}{2M^*} = \frac{2k(k + p_F \cos \alpha)}{M^*} \ , \tag{5.21}$$

where α is the angle between $\vec{p}_i$ and $\vec{k}$ and $p_i = p_F$ has been used. Assuming, again realistically (see below),

$$k \propto p_F \ , \tag{5.21a}$$

we get

$$\omega_i - \omega_f = \Delta E = 2(p_F^2/M^*)\ (1 + \cos \alpha) \tag{5.22}$$

$$= 210\ (1 + \cos \alpha)\ \text{MeV} \ , \tag{5.22a}$$

assuming[46] $p_F = 1.33\ \text{fm}^{-1}$, $M^* = M/1.43$. This result is enormous, and shows that backward scattering of the pion simply cannot happen, unless k decreases in the process. In any case, a large amount of energy is transferred from the pion to the nucleon.

We conclude that the assumption of constant k cannot hold indefinitely to arbitrarily small values of the pion energy. This is in accord with the discussion in Sec. VI. At low energies, k^2 is negative, because the repulsive s-scattering dominates. Above about 30 MeV, s-scattering is no longer so important, and k^2 increases rapidly. Because of the Kisslinger syndrome, k^2 is apt to become very large (Figs. 8 and 9) unless it is cut down by the "damping" term, i.e., the imaginary term in Eq. (3.29). This term, however, cannot be large unless k is large, so on every argument k must be large. On the other hand, near the resonance, Re k should be fairly close to the free-particle value k_0, because near the resonance we expect Π(k) to be purely imaginary. We therefore adopt the following model.

We assume that k is constant and has the value k_r (resonance value of k_0) above a certain critical $\omega = \omega_c$. Near ω_c, k changes very rapidly from 0 to k_r, thus

$$\begin{aligned} k &= 0 \ , \ \omega < \omega_c \approx 1.2\mu \\ k &= k_r = 2.18\mu \ , \ \omega > \omega_c \end{aligned} \quad . \tag{5.23}$$

The estimate $\omega_c = 1.2\ \mu$ is based on the numbers of Sec. VI. Now it happens accidentally that

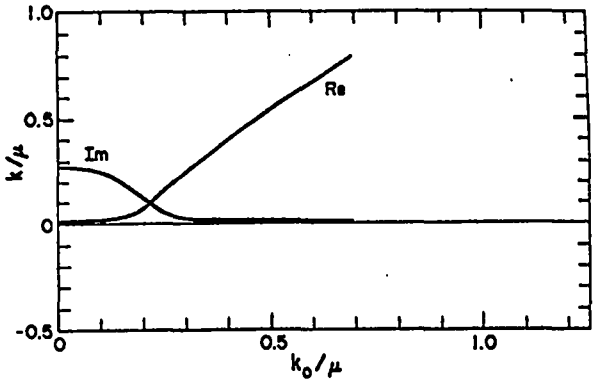

Fig. 8. Real and imaginary part of k/μ, for small k_0, as a function of k_0/μ. Theory of Sec. VI has been used, no damping for quasi-elastic scattering is considered. For $\rho = \frac{1}{2}\rho_0$.

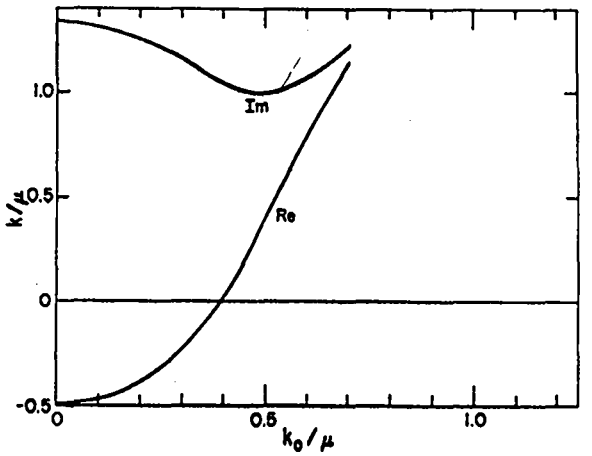

Fig. 9. Real and imaginary part of k/μ, for small k_0, as a function of k_0/μ. Theory of Sec. VI has been used, no damping for quasi-elastic scattering is considered. For $\rho = \frac{1}{4}\rho_0$.

$$k_r = 2.18\mu = 1.53 \text{ fm}^{-1} \approx p_F \quad . \tag{5.24}$$

Thus in this model the assumptions in Eqs. (5.21) to (5.22a) are justified.

This model is calculated in Appendix B with the approximate result given in Eq. (B.21); thus the "damping integral" L is

$$\begin{aligned} L &\equiv \int_{-1}^{\lambda} k'^3\, \omega'\, d(\cos\chi') \\ &= k_r^2\, (M^*/2P)\, (\omega_i - \omega_c) \\ &\times \left[\left(2 + \frac{M^*}{M+\omega_i}\right)\left(\omega_i - \frac{P^2}{2M^*}\right) + \left(1 - \frac{M^*}{M+\omega_i}\right)\omega_c \right]. \end{aligned} \tag{5.25}$$

Using $\omega_i = 2.4\ \mu$ (resonance energy),

$\omega_c = 1.2\ \mu$ (30 MeV kinetic energy),

$$\begin{aligned} P &= \left(k_r^2 + p_F^2\right)^{\frac{1}{2}} = 2.88\ \mu \\ M^* &= M/1.35 = 5.0\ \mu \quad , \end{aligned} \tag{5.26}$$

we find

$$L = 22.5\ \mu^4 \quad , \tag{5.27}$$

which agrees with a more accurate evaluation. (See Appendix B.)

In the elementary theory of Sec. III, taken literally, we would have instead

$$L_0 = 2k_i^3\omega_i = 49.6\ \mu^4 \quad . \tag{5.28}$$

But even in an elementary theory, we should evaluate k and ω in the center-of-mass system of pion and nucleon. Using the well-known relativistic formula

$$\left(M + \omega'\right)^2 = M^2 + \mu^2 + 2M\,\omega_i \quad , \tag{5.29}$$

and taking again $\omega_i = 2.4\ \mu$, we have

$$\omega_i' = 2.14\ \mu, \quad k_i' = 1.89\ \mu$$

so the "damping integral" is

$$L_0' = 2k_i'^3\omega_i' = 28.9\ \mu^4 \quad . \tag{5.30}$$

As mentioned, our theory gives $L = 22.5\ \mu^4$. Therefore the Pauli principle and energy loss together reduce the "damping integral" L by a factor of 1.28. The Pauli principle alone would reduce L by a factor of 1.14. So we get the surprising result that the nuclear recoil does not have much influence on the damping integral, except that which is already present for free nucleons.

At lower energy ω_i, the L will be more strongly reduced [compare Eq. (5.25)] provided k_i remains high as it seems to do (see Sec. VII). Analytically, L depends on $k_i = k_r$ only quadratically, while the elementary L' goes as k_i^3. Instead, L depends on the energy of the incident pion, as $\omega_i - \omega_c$. The factor ω_i' in Eq. (5.30) is essentially replaced by one-half of the square bracket in Eq. (5.25).

In doing calculations with Eq. (3.29), we should substitute in the bracket

$$k_\omega^3 \to L/2\omega' \quad . \tag{5.31}$$

Since L depends on P, an integration over P must be performed [see Eqs. (4.27) and (4.28)]. For an approximation (Sec. VII) we have used Eq. (5.26) as an average value of P.

VI. LOW-ENERGY BEHAVIOR

The lowest energy pions are the π^- in atomic orbits in pionic atoms. These have been most carefully investigated in a series of papers, theoretical ones[5,9] beginning with that of the Ericsons, and experimental ones, summarized by Backenstoss[10] and later by Tauscher.[10] If we restrict ourselves to nuclei having equal numbers of neutrons and protons, the behavior of low-energy pions can be described by just four parameters,

b_0 = scattering of s-wave pions,

c_0 = scattering of p-wave pions,

B_0 = probability of capture of s-wave pions by 2 nucleons, by the process in Eq. (1.11), and

C_0 = same for p-wave pions.

Assuming that these parameters do not depend on the nuclear density, we have

$$k^2 = k_0^2 + 4\pi\rho[\, b_0 + iB_0\rho + k^2(c_0 + iC_0\rho)] \quad . \tag{6.1}$$

In Ref. 5, the result is presented somewhat differently, e.g., the last term in Eq. (6.1) is replaced by

$$-\nabla \cdot (c_0\rho + iC_0\rho^2)\,\nabla \quad . \tag{6.2}$$

For constant ρ, this reduces to $k^2(c_0\rho + iC_0\rho^2)$. In actual calculations of π-mesic atoms, it is essential to use Eq. (6.2)(see Sec. VIII). We have also omitted such factors as $1 + \mu/M$ in Eq. (6.1).

The constants iB_0, iC_0 in Eq. (6.1) are denoted by B_0, C_0 in Ref. 5. These quantities are in fact complex, not purely imaginary. But the imaginary part is, in practice, the most important one since b_0 and c_0 are purely real. The imaginary part gives the capture probability which the Ericsons have calculated from the elementary processes. The real parts are the corresponding "energy shifts" which might, in principle, be derived from the imaginary parts by some dispersion theory, but are in practice essentially unknown.

The scattering amplitude c_0 has been derived by the Ericsons, for zero kinetic energy, from the four measured p-wave phase shifts, δ_{11}, δ_{13}, δ_{31}, and δ_{33}, and the result is in good agreement with π-mesic atoms. The constant b_0 is, in first approximation, given by the weighted average of the s-wave phase shifts

$$b_0^0 = (\delta_1 + 2\delta_3)/3k \quad . \tag{6.3}$$

But this quantity, as derived from measured π-nucleon scattering, is almost exactly zero. The Ericsons point out a number of corrections of which the most important is due to the correlations of nucleons due to the antisymmetry of the wave function. This effect makes b_0 effectively negative, but comparison with experiment is not easy.

The experiments consist of measurements on π-mesic atoms.[10] The shift of the x-ray levels relative to the pure Coulomb field gives b_0 (mostly from the 1s state) and c_0 (mostly from 2p). The width of the levels gives B_0 (from 1s) and C_0 (mostly from 2p). The most recent experiments give[10]

$$\begin{aligned} b_0 &= -\,0.0293 \pm 0.0005\ \mu^{-1}, \\ c_0 &= \ \ 0.227 \ \pm 0.008\ \ \mu^{-3}, \\ B_0 &= \ \ 0.0428 \pm 0.0015\ \mu^{-4}, \text{ and} \\ C_0 &= \ \ 0.076 \ \pm 0.013\ \ \mu^{-6}. \end{aligned} \tag{6.4}$$

The precision, especially in b_0 and B_0, is impressive.

We have used the values in Eq. (6.4), together with the formula in Eq. (6.1), to calculate k^2 as a function of ρ and k_0^2. In doing this, we have made three assumptions, namely,

(a) The constants b_0, etc., are independent of energy, which seems justified up to kinetic energies of about $0.2\ \mu = 28$ MeV.

(b) The "damping terms" due to quasi-elastic scattering, discussed in Secs. III-V, are negligible. This is justified because it is difficult for the pion to lose energy in the laboratory system.

(c) We have taken into account the Lorentz-Lorenz correction [Eq. (2.12)], i.e., we have replaced the last term in Eq. (6.1) by

$$\frac{4\pi\rho(c_0 + iC_0\rho)}{1 + (4\pi/3)\rho(c_0 + iC_0\rho)}\,k^2 \quad . \tag{6.5}$$

The result was given in Figs. 8 and 9 for $\rho = \frac{1}{2}\rho_0$ and $\rho = \rho_0$. In the latter case, it is

remarkable that Re k is negative for $k_0 < 0.4\ \mu$, $\omega < 1.07\ \mu$. Even as high as $\omega = 1.25\ \mu$, we have Re k = Im k, i.e., k^2 is purely imaginary! This means that the s-state repulsion dominates over the p-state attraction, even though b_0 is very small. At the same time, Re k and Im k both tend to be considerably larger than k_0, because of the "Kisslinger syndrome".

For $\rho = \frac{1}{4}\rho_0$, both parts of k are much better behaved, as might be expected.

Because of the behavior of k, we have chosen (in Appendix B) to set the "critical energy" $\omega_c = 1.2\ \mu$ because only above this ω_c does Re k become appreciable and positive.

VII. NUMERICAL CALCULATIONS

We did numerical calculations using our theory, with the following approximations.

1. In Eq. (5.25), P was replaced by its root mean square average,

$$P_{av} = \left(k_r^2 + p_f^2\right)^{\frac{1}{2}} \quad . \tag{7.1}$$

2. In the bracket in Eq. (5.25), the term P^2/M^* was neglected.
3. The capture of pions, terms B_0 and C_0 of Sec. VI, has been neglected.
4. The Lorenz-Lorentz effect [Eq. (2.12)] has been neglected.
5. The effect of the small phase shifts has been neglected (see the end of this section).

We have done three separate calculations. In one of these (A) we used

$$\omega_c = 1.2\ \mu,\ \omega_r = 2.14\ \mu \quad , \tag{7.2a}$$

i.e., we used the center-of-mass resonance energy. In the two others we used the laboratory resonance energy, and in calculations (B) and (C), respectively, we set

$$\omega_c = 1.2\ \mu,\ \omega_r = 2.40\ \mu \quad , \tag{7.2b}$$

and

$$\omega_c = 1.0\ \mu,\ \omega_r = 2.40\ \mu \quad . \tag{7.2c}$$

Thus in (A) and (B), we set the "critical" energy $\omega_c = 1.2\ \mu$, while in (C) we set it equal to μ. The latter is, of course, the lowest possible energy to which the pion may be degraded; the former corresponds approximately to the results of Sec. VI.

We introduce the abbreviations and numerical values.

$$x = \omega/\mu,\ y = k/\mu,\ a = \omega_r/\mu,\ b = \omega_c/\mu \quad , \tag{7.3a}$$

$$B = \frac{4}{3} f^2 a,\ f^2 = 0.080 \quad , \tag{7.3b}$$

$$C = 8\pi\rho_0\mu^{-3}C_3,\ C_3 = 2/3,\ C = 7.8 \quad , \tag{7.3c}$$

$$r = \rho/\rho_0,\ p_F/\mu = d_0 r^{\frac{1}{3}} \quad , \tag{7.3d}$$

$$d_0 = p_{F0}/\mu = 1.90 \quad , \tag{7.3e}$$

$$\frac{M^*}{\mu} = \frac{M/\mu}{1 + 0.53r^{\frac{2}{3}}} \quad , \tag{7.3f}$$

$$\phi = \frac{M^*}{2\mu P}\left[2 + \left(\frac{M^*}{M + \omega_i}\right)\left(\omega_i - \frac{P^2}{2M^*}\right) + \left(\frac{M^*}{M + \omega_i}\right)\omega_c\right] \quad ,$$

and

$$P = \left(y^2 + d_0^2 r^{\frac{2}{3}}\right)^{\frac{1}{2}} \quad . \tag{7.4}$$

Then the fundamental Eqs. (2.2), (2.10), (3.31), and (5.31) give

$$\frac{d}{dr}(y^2) = \frac{BCy^2}{x(a - x) - iBy^2\phi} \quad , \tag{7.5}$$

and

$$y^2(r = 0) = x^2 - 1 \quad . \tag{7.5a}$$

Note that Eq. (7.5) is a differential equation for y(r), at constant x.

The results are given in Figs. 10-12. All of these give the real and imaginary part of k as a function of k_0, and all of them refer to cases B and C [Eqs. (7.2 b,c)] i.e., $\omega_r = 2.4\ \mu$. The resonance momentum is thus

$$k_{0r} = 2.18\ \mu \quad . \tag{7.6}$$

The dashed curves are for case C, the solid ones for case B. The three figures refer, respectively, to $\rho = \frac{1}{2}\rho_0$, ρ_0 and $2\rho_0$.

For all three densities, the internal wave number Re k rises initially much faster than k_0 as is to be expected for an attractive potential, then

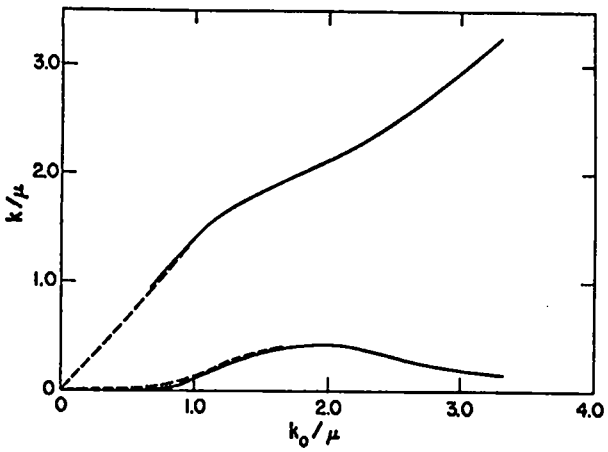

Fig. 10. Real and imaginary part of k/μ, with damping for quasi-elastic scattering. Model of Appendix B has been used. Solid curve: $\omega_c = 1.2\ \mu$, dashed curve: $\omega_c = 1.0\ \mu$. $\rho = \frac{1}{2}\rho_0$.

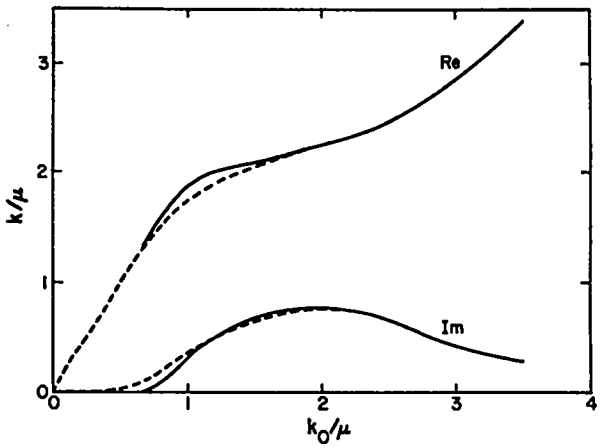

Fig. 11. Real and imaginary part of k/μ, with damping for quasi-elastic scattering. Model of Appendix B has been used. Solid curve: $\omega_c = 1.2\ \mu$, dashed curve: $\omega_c = 1.0\ \mu$. $\rho = \rho_0$.

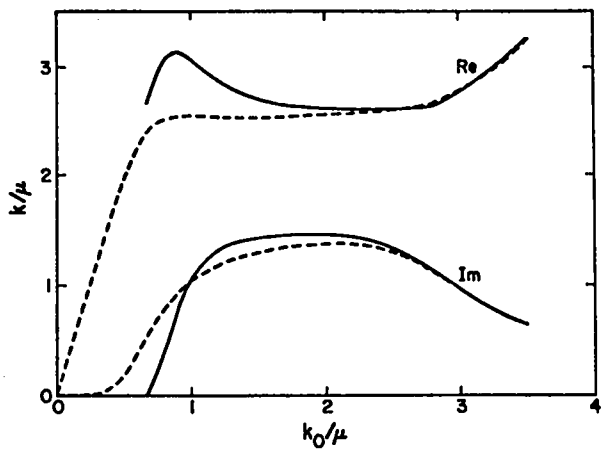

Fig. 12. Real and imaginary part of k/μ, with damping for quasi-elastic scattering. Model of Appendix B has been used. Solid curve: $\omega_c = 1.2\ \mu$, dashed curve: $\omega_c = 1.0\ \mu$. $\rho = 2\rho_0$.

falls below k_0 for energies above the resonance. For $\rho = 2\rho_0$, the effect is most pronounced. In fact, k reaches a maximum at $k_0/\mu = 0.9$ (50-MeV kinetic energy), and then decreases with increasing energy! This is just the opposite of the elementary, Kisslinger theory [Eq. (1.6)] in which k rises monotonically with k_0 (provided the denominator of Eq. (1.6) stays positive). At the lower densities, including $\rho = \rho_0$, the result is not so extreme. At $\rho = 1.5\rho_0$, the inside k stays essentially constant for k_0/μ from 1.0 to 2.6. This is the assumption we made in Sec. V and Appendix B. For $\rho = \rho_0$, this assumption is still quite fair. In case A [Eq. (7.2a)], the inside k is constant (over a wide range of k_0) for $\rho = \rho_0$.

Another way to look at the result is to regard $k^2 - k_0^2$ as a potential U. This potential, even for ρ as low as $0.25\ \rho_0$, will decrease (in absolute value) with increasing energy, while the elementary theory, Eq. (1.4), will have it increase monotonically. If we replace k^2 by k_0^2 in Eq. (1.4), then the second term of Eq. (1.4) is $k_0^2 a_1(\omega)$. Both factors in this expression increase with energy.

In all our results, k remains quite moderate, and the "Kisslinger syndrome" of Eq. (1.6) has been completely eliminated. Moreover, Eq. (1.6) would predict that somewhere below the free-nucleon resonance we have

$$\text{Re } 4\pi\rho a_1 = 1 \qquad (7.7)$$

so that the denominator becomes purely imaginary. Here k^2 would be purely imaginary, and hence

$$\text{Im } k = \text{Re } k \quad . \qquad (7.7a)$$

In our theory, the imaginary part of k always remains much smaller than the real part; e.g., for $\rho = \rho_0$, nearly everywhere

$$\text{Im } k < \tfrac{1}{3}\ \text{Re } k \quad . \qquad (7.7b)$$

The maximum value is

$$(\text{Im } k)_{\max} = 0.8\ \mu = 0.56\ \text{fm}^{-1} \qquad (7.8)$$

so that the absorption length of a pion is about 0.9 fm. Although this is small, it is not nearly as small as would follow from free-nucleon cross sections,[25] thus solving the problem of excessive absorption cross section mentioned in Sec. I. The cross section corresponding to Eq. (7.8) is 70 mb. Im k stays high over a wide range of energies; e.g., at $\rho = \rho_0$, we have

$$\text{Im } k > 0.2\ \mu \tag{7.8a}$$

for $k_0 > 0.92\ \mu$; i.e., $\omega - \mu > 50$ MeV. A pion going through the center of a ^{12}C nucleus traverses a distance $2R = 5.2$ fm; then Im $k = 0.2\ \mu$ corresponds to an absorption of the pions to ½ of the incident intensity.

Roughly speaking, the calculations indicate

$$\text{Im } k = \phi(\omega)\rho/\rho_0 \quad , \tag{7.9}$$

where ρ is some function of the energy. This means that the apparent absorption cross section is roughly independent of ρ. The maximum of Im k occurs roughly at the free-nucleon resonance momentum [see Eq. (7.6)]. Also, again very roughly,

$$\text{Re } k - k_0 = \chi(\omega)\rho/\rho_0 \quad . \tag{7.10}$$

Of particular interest is the point at which

$$\text{Re } k = k_0 \quad . \tag{7.11}$$

As discussed in detail in Sec. X, this is the best definition of the position of the resonance in nuclear matter. From Figs. 10-12 and similar curves, we read the values of k_0 at which Eq. (7.11) is fulfilled. These are given in Table I under the heading "33 alone." It is seen that the "resonance" as defined by Eq. (7.11) is shifted to slightly higher values of k_0 as compared with k_r [Eq. (7.6)]. Roughly we find

$$k_{n33} = k_r + 0.24\ \mu\ \rho/\rho_0 \quad . \tag{7.12}$$

Small Phases

However, we have so far only taken into account the scattering in the 33-state. The "small phase shifts" δ_1, δ_3, δ_{11}, δ_{13}, and δ_{31} also contribute to the forward scattering and hence to the self-energy Π. We write simply

$$f(k) = f_{33}(k) + f_{sm}(k) \quad , \tag{7.13}$$

where f_{sm} denotes the contribution of the small phase shifts. This expression neglects the fact that f_{sm} will change the relation between k and k_0 by Eqs. (2.2) and (2.11), and hence will change f_{33} for a given ω, [see Eq. (3.31)]; but we believe this correction is small. Now f_{sm} is independent of ρ, hence its contribution to Eq. (2.11)

$$\Pi_{sm} = 4\pi\rho f_{sm} \quad . \tag{7.14}$$

Therefore we have [see Eq. (2.2)], for any given ω,

$$k^2 = k_{33}^2 + 4\pi\rho f_{sm} \quad , \tag{7.15}$$

where k_{33} is the k calculated by our previous theory, i.e., taking only the 33-scattering into account. Now f_{sm} is essentially real, hence

$$\begin{aligned} \text{Re}(k^2) &= \text{Re}(k_{33}^2) + 4\pi\rho f_{sm} \\ \text{Im}(k^2) &= \text{Im}(k_{33}^2) \quad . \end{aligned} \tag{7.16}$$

Algebra yields

$$\text{Re } k - \text{Re } k_{33} = 2\pi\rho \frac{\text{Re } k}{|k|^2} f_{sm} \quad . \tag{7.17}$$

Now we are interested in the resonance momentum in nuclear matter, k_n, for which Re $k = k_0$. This will occur when

$$\text{Re } k_{33} = k_0 - 2\pi\rho f_{sm} \text{Re } k/|k|^2 \quad . \tag{7.18}$$

Now Re $k = k_0$, and Im $k \ll$ Re k, especially when $\rho \ll \rho_0$ [Eq. (7.9)], so

$$\text{Re } k_{33} \approx k_0 - 2\pi\rho f_{sm}/k_0 \quad . \tag{7.19}$$

We have obtained the small phase shifts from the tables of Herndon et al,[52] in particular from

their "Cern theoretical fit," pp. 80-81. The forward scattering amplitude by an "average nucleon" is

$$f_{sm} = \tfrac{2}{3} f_{sm}(T = \tfrac{3}{2}) + \tfrac{1}{3} f_{sm}(T = \tfrac{1}{2}) \qquad (7.20)$$

$$f\ (T) = \sum_{\ell j} (j + \tfrac{1}{2})\ \delta\ (T, j)\ /\ k_0 \quad , \qquad (7.21)$$

where we have set $\sin \delta\ e^{i\delta} = \delta$. We have included all phases given in the table of Herndon et al. At the resonance energy E = 1236 MeV, we have[53]

$$f_{sm} = -\ 0.095/\mu \quad . \qquad (7.22)$$

Taking the correction term in Eq. (7.19), $k_0 = 2.18\mu$, $\rho_0 = 0.467\ \mu^3$, we get

$$\mathrm{Re}\ k_{33} = k_0 + 0.128\ \rho/\rho_0 \quad . \qquad (7.23)$$

We may then again use Figs. 10-12 to find the value of k_0 at which Eq. (7.23) is fulfilled. This is listed in Table I under "complete."

Comparing the "complete" column with the "33 alone", we see that the small phases shift the resonance down. This is to be expected because f_{sm} is negative [Eq. (7.22)], i.e., the small phases correspond to a repulsive potential. Therefore to get a resultant potential of zero required for resonance, the 33-term must contribute an attractive potential, i.e., the energy is below the energy of the resonance calculated with 33 alone. The resulting resonance momentum k_n, according to Table I, is slightly below the free-nucleon resonance as long as $\rho \leq \rho_0$.

TABLE I

POSITION[a] OF RESONANCE, DEFINED BY RE k = k_0

ρ/ρ_0	33 Alone	Complete
0	2.18	2.18
0.25	2.25	2.10
0.5	2.30	2.12
0.75	2.35	2.14
1.0	2.40	2.19
1.5	2.50	2.25
2.0	2.65	2.34

[a] We give k_0/μ at resonance

The data are somewhat scattered, and we estimate that the correct k_n for $\rho = \frac{1}{2}\ \rho_0$ may be 2.10 ± 0.02 corresponding to a downward shift of the resonance by about 11 ± 3 MeV.

Different Models

Figures 10-12 show that models B and C give very similar results, so the answer does not depend much on the assumed value of ω_c, which is satisfactory. Of course, the general model we used, constant k from ω_c to ω_i (see Sec. V), is fully justified only for $\rho = 1.5\ \rho_0$ where the resulting $k(\omega)$ is consistent with the assumption. For $\rho = 1.0$ or $2.0\ \rho_0$, the assumption should still be quite good; for smaller ρ, it is poor, and the calculation should be repeated with more realistic assumptions about $k(\omega)$.

Our calculations show that model A, in which the resonance momentum is lower, gives results for any given ρ which are like those for models B and C at a somewhat higher ρ. The upward shift of the resonance with the 33-scattering alone is therefore greater than given in Table I for any given ρ. The "complete" shift, including the small phases, is closer to zero at low density.

VIII. NUCLEAR SURFACE

In his first paper, Kisslinger[1] pointed out that for finite nuclei, where the density changes with position, the potential proportional to k^2 should be replaced as follows [see Eq. (1.4)].

$$k^2\ a_1(\omega)\rho \rightarrow -a_1(\omega)\nabla\cdot(\rho\nabla) \quad . \qquad (8.1)$$

This is necessary to make the Schrodinger equation self-adjoint. The same expression has been used by the Ericsons[5] and others, and has proved successful in the treatment of pionic atoms. It is often called a "nonlocal potential."

Krell and Ericson[9] have shown how Eq. (8.1) can be transformed to a local potential. Instead of the radial wave function ϕ_ℓ, they introduce[54]

$$u_\ell = \left[1 - \alpha(r)\right]^{\frac{1}{2}} r\phi_\ell(r) \quad , \qquad (8.2)$$

where approximately

$$\alpha(r) = 4\pi\rho(c_0 + i\rho C_0) \quad . \qquad (8.3)$$

Then u_ℓ obeys a Klein-Gordon equation with an effective local potential U. For reasons to be discussed below, we shall only reproduce the parts of this equation which are of first order in ρ. We have

$$\frac{d^2u_\ell}{dr^2} - \frac{\ell(\ell+1)}{r^2}u_\ell + \left[\frac{k_0^2}{1-\alpha(r)} - q(r) - U_g(r)\right]u_\ell = 0 \quad , \tag{8.4}$$

where k_0 is the free-space wave number, except that it includes the effect of the Coulomb potential, c is the effect of the s-wave interaction [arising from b_0 and B_0 in Eq. (6.1)], and U_g (g for gradient) arises from the p-wave interaction,

$$U_g = -2\pi c_0\nabla^2\rho = -2\pi c_0\left(\frac{d^2\rho}{dr^2} + \frac{2}{r}\frac{d\rho}{dr}\right) \quad . \tag{8.5}$$

Another effect of the p-wave interaction is the term $\alpha(r)$ in the denominator[56] with k_0^2. Expanding this term in powers of ρ, we get

$$k_0^2 + k_0^2\alpha(r) + \ldots = k_0^2 + 4\pi c_0 k_0^2\rho + \ldots \tag{8.6}$$

in which again higher powers of ρ are omitted. Collecting all terms arising from the p-wave interaction, i.e., proportional to c_0, we get the equivalent p-wave potential

$$U_p = -4\pi c_0(k_0^2\rho + \tfrac{1}{2}\nabla^2\rho) \quad . \tag{8.7}$$

This result has also been derived in a simpler, but less convincing, manner by Wilkin.[57] Clearly, Eq. (8.7) is dimensionally correct.

We have neglected all higher powers in ρ because our theory deviates from that of Krell and Ericson in order ρ^2 and higher, due to the "damping term" which has been discussed at length in Secs. III-V. But in the lowest order[58] in ρ, the theories agree. As shown in Sec. IX, the most interesting region of the nucleus for <u>elastic</u> scattering is indeed the region of low density, perhaps 1/4 to 1/10 ρ_0. In this density region, the Krell-Ericson theory is justified,[59] but also its first-order term in ρ is sufficient.

The extra term ∇^2 is, in general, not large for scattering problems. As an example, let us take a density distribution of the "Fermi" type which has been used so much in the analysis of the Stanford electron scattering experiments, namely,

$$\rho = (1 + e^x)^{-1}, \quad x = \frac{r - R}{a'} \quad . \tag{8.8}$$

Experiments show that

$$a' = 0.55 \text{ fm} = 0.4\ \mu^{-1} \quad . \tag{8.8a}$$

In this case, one calculates that

$$\frac{\nabla^2\rho}{\rho} = \frac{1}{a'^2(1+e^{-x})}\left(\frac{1-e^{-x}}{1+e^{-x}}\right) - \frac{2a'}{r} \quad . \tag{8.9}$$

Then, arbitrarily taking $\rho = \rho_0/7$ (see Sec. IX) and R = 2.58 fm (correct for ^{12}C), we find

$$\frac{\nabla^2\rho}{\rho} = 1.15 \text{ fm}^{-2} = 2.35\ \mu^2 \quad . \tag{8.10}$$

In most experimental cases, $2k_0^2$ is larger than this, so that the $\nabla^2\rho$ term in Eq. (8.7) is only moderately important. But for pionic <u>atoms</u>, the $\nabla^2\rho$ dominates by far, and this fact has been used by Krell and Ericson.[9]

Krell and Ericson point out that the $\nabla^2\rho$ term, plus other terms of higher order in ρ, represent a "dipole" interaction at the surface of the nucleus. For large r, $\nabla^2\rho > 0$, as is clear from the form used in Eq. (8.8) so that the potential U_g is attractive, whereas for r < R, both terms on the right of Eq. (8.5) are negative, and $U_g > 0$. This argument is not changed (qualitatively) by the terms of higher order in ρ.

With increasing energy, $a_1(\omega)$ in Eq. (8.1) becomes complex, and at resonance, is purely imaginary. Therefore, U_g is also imaginary. Then the words "attractive potential" must be replaced by "additional absorption" and "repulsive potential" by "reduced absorption." But in this case, $2k_0^2 = 9.5\mu^2$, which is large compared to Eq. (8.10).

We have followed the ideas of previous writers on the effect of density variations near the nuclear surface. It would be better to have a theory specifically designed for a finite nucleus. We have

not found such a theory, therefore we use a local density approximation (LDA). The discussion of this section shows that this approximation may be good, even in the surface region, as soon as k_0 is substantial.

IX. EIKONAL APPROXIMATION

We shall make the local density approximation (LDA), i.e., we assume that at every point in a finite nucleus

$$k = k(\omega,\rho) \quad , \tag{9.1}$$

where ρ is the local density at that point. The considerations of Sec. VIII show that this should be a fair approximation if ω is not too small.

We then consider the pion wave going through the nucleus at an impact parameter b. This is the eikonal approximation. It neglects the quantization of angular momentum, and thus assumes, in effect,

$$k_0 b \gg 1 \quad . \tag{9.2}$$

At the observed resonance in ^{12}C, 175 MeV, $k \approx 2\ \mu$, while the most important values of b are about 2.2 μ^{-1} (see Table II). Thus,

$$k_0 b \approx 4.5 \tag{9.2a}$$

TABLE II

RELEVANT DENSITY $\rho(b_1)/\rho_0$ FOR PION SCATTERING, AND CORRESPONDING IMPACT PARAMETER b_1

Nucleus	^{12}C		Pb
$Im\phi(\omega)$	0.9 μ	0.5 μ	0.9 μ
k_0/μ	2.0	1.25	2.0
Kin. Energy (MeV)	175	85	175
a(fm)	0.63	0.82	0.68
R(fm)	2.58	2.58	6.64
b_1(fm)	3.18	2.49	7.49
$(2\pi a b_1)^{1/2}$	3.55	3.58	5,67
$k_0 b_1$	4.5	2.2	10.5
$\rho(b_1)/\rho_0$	0.31	0.54	0.20

so that Eq. (9.2) is quite well fulfilled. In an angular momentum analysis, the most important will be

$$\ell_1 \approx k_0 b - \tfrac{1}{2} \approx 4 \quad . \tag{9.3}$$

If the incident wave is

$$\psi_0 = e^{ik_0 z} \quad , \tag{9.3a}$$

the wave which has gone through the nucleus at impact parameter b, will be

$$\psi = \psi_0 \exp i\int_{-\infty}^{\infty} dz \;\; k[\rho(r)] - k_0 \tag{9.4}$$

where

$$r = (z^2 + b^2)^{1/2} \quad . \tag{9.4a}$$

Now in the interior of the nucleus, $\rho \approx \rho_0$, and here the imaginary part of k is very large; e.g., near the resonance (Fig. 11)

$$Im\ k \approx 0.8\ \mu \quad . \tag{9.5}$$

Taking the nucleus ^{12}C for which most experiments have been done, the radius is

$$R = 1.12\ A^{1/3} = 2.58\ fm = 1.80\ \mu^{-1} \quad . \tag{9.6}$$

Hence for b = 0, the absorptive part of the integral in Eq. (9.4) is

$$2R\ Im\ k = 2.88 \quad . \tag{9.7}$$

Thus, the amplitude of this wave is reduced by about a factor of 20, the intensity by a factor of 300. Even a nucleus as light as ^{12}C is black for pions going through its interior.[60] As Fig. 11 shows, this result does not change substantially if k_0/μ varies between the limits 1.2 and 2.8 (kinetic energy between 80 and 280 MeV).

The most interesting values of b are those for which

$$|\psi/\psi_0| \approx \tfrac{1}{2} \quad . \tag{9.8}$$

This follows from the familiar argument of eikonal theory. Consider a plane $z = z_1$ where z_1 is somewhat larger than R. Then consider

$$S(b) = \Psi(b,z_1) - \Psi_0(b,z_1) \tag{9.9}$$

as a source of spherical waves corresponding to the Huyghens construction. (This argument can be improved by considering a closed surface surrounding the nucleus, e.g., a sphere of radius $r_1 > R$, and giving $\psi - \psi_0$ on that surface.) Clearly, for large b, Eq. (9.9) is zero; for small b, it is not zero but uninteresting, viz., $-\psi_0$. The interesting values are those near the condition of Eq. (9.8).

It is clear from Eqs. (9.7) and (9.8) that the important values of b are large, so that the pion goes only through the surface of the nucleus and encounters only low density. In this region, we make the simple assumption that

$$\rho(r) = \rho(b) \exp (b-r)/a \quad . \tag{9.10}$$

We approximate Eq. (9.4a) by

$$r = b + z^2/2b \quad , \tag{9.10a}$$

which turns out to be sufficiently accurate[61] because $b \gg a$. We assume, in accord with Eqs. (7.9) and (7.10),

$$k(\rho) - k_0 = \phi(\omega)\, \rho/\rho_0 \quad , \tag{9.11}$$

where ϕ is a complex function of the energy ω.[62] Then the integral in Eq. (9.4) becomes

$$X(b) = i\phi(\omega)\, \rho(b)\, \rho_0^{-1}\, (2\pi a\, b)^{\frac{1}{2}} \quad . \tag{9.12}$$

To determine a, we use two approaches.

(1) The Stanford distribution [Eq. (8.8)]. We shall find $\rho(b)/\rho_0 \approx 0.3$. We may then define a as the distance in which ρ/ρ_0 decreases from 0.3 to 0.3/e; thus

$$a = 0.68 \text{ fm} \quad . \tag{9.13a}$$

(2) We may take the binding energy of the last proton in ^{12}C which is 15.9 MeV, and calculate the asymptotic decrease of the Schrodinger wave function. This gives

$$a = \tfrac{1}{2}\, (20.7/15.9)^{\frac{1}{2}} = 0.57 \text{ fm} \quad . \tag{9.13b}$$

We shall take the average between Eqs. (9.13a) and (9.13b),

$$a = 0.63 \text{ fm} \quad . \tag{9.13}$$

For ϕ in Eq. (9.12), we use the curve similar to Fig. 10 but calculated for $\rho/\rho_0 = 0.25$. Then, at resonance, we have

$$\text{Im}\phi(\omega) = 0.9\ \mu \tag{9.14}$$

We are interested in the value b_1 of b for which Eq. (9.8) is true [see Eq. (9.12)].

$$\text{Re}\ X(b_1) = -\ln 2 \quad . \tag{9.15}$$

The results for b_1 and $\rho(b_1)/\rho_0$ are given in Table II, together with other quantities occurring in the analysis. We have considered ^{12}C and Pb at the resonance and at an energy where the effective absorption [Eq. (9.14)] is 0.5 μ. This energy, from our calculations, is 85 MeV.

The nucleon density at which Eq. (9.8) is true, $\rho(b_1)/\rho_0$, is 0.31 for ^{12}C at resonance, only 0.20 for Pb, and 0.54 for ^{12}C at 85 MeV. So certainly at resonance, the density which is most relevant for pion scattering is far below nuclear matter density. This was recognized by Dover and Lemmer[15] who, in their Fig. 24, plot the cross sections for π-scattering by ^{12}C as functions of the energy, assuming various densities for the nucleus: $\rho = 0.16\ \mu^3 \approx 0.34\ \rho_0$ gives the best, though not perfect, agreement with experiment.

The fact that $\rho(b_1) \ll \rho_0$ is of very great importance. It means that for the calculation of elastic scattering, any theory is applicable which is correct at these low densities. At these densities, we are far from the Kisslinger singularity so the Kisslinger theory, [Eq. (1.6)], is alright, as well as the "local" modification thereof [Eq. (8.7)]. The first-order term in ρ, used in Sec. VIII, is then also alright. This explains the considerable success achieved in calculations of differential cross sections using the Kisslinger and related theories.[12,57,63]

We must not conclude from this success, however, that the Kisslinger theory remains correct at higher densities. The region of higher density is simply not explored by the elastic scattering because the nucleus is black. In Sec. XII we consider experiments to explore the high-density region.

The other data in Table II are also interesting. For resonance pions the "relevant" impact parameter b_1 is considerably greater than the usual nuclear radius. This explains that the cross section is substantially greater than geometrical; e.g., at 180 MeV, the observed reaction cross section of ^{12}C is 425 mb, corresponding to an "effective radius" of 3.68 fm, which is very close to our $b_1 = 3.55$. Similarly, the elastic scattering corresponds to a large effective nuclear size. At 85 MeV, the effective b_1 is only about R, which is in accord with the decreased reaction cross section.

The expression $(2\pi\ a\ b_1)^{\frac{1}{2}}$ gives the effective length of the chord traversed by the pion. The mean density along this chord is

$$\frac{\int \rho\ \rho\ dz}{\int \rho\ dz} = \frac{\rho(b)}{\sqrt{2}} \quad . \tag{9.16}$$

The value of $k_0 b_1$ gives a guide as to the most important value of ℓ in scattering, see Sec. XI.

Deviations from the local density approximation (LDA) may be discussed in terms of the local form of the p-wave potential [Eq. (8.7)]. In rough approximation, we might replace ρ by

$$\rho_{eff} = \rho + \nabla^2\rho/2\ k_0^{\ 2} \quad . \tag{9.17}$$

If we evaluate Eq. (8.9) at $\rho/\rho_0 = 0.25$, corresponding approximately to $\rho(b_1)$ of Table II, we find instead of Eq. (8.10), that

$$\nabla^2\rho/\rho = -0.39\ fm^{-2} = 0.80\mu^2 \quad . \tag{9.18}$$

Then at resonance ($k_0/\mu = 2$), Eq. (9.17) gives $\rho_{eff} = 1.10\ \rho$. The last line in Table II should then refer to ρ_{eff}, so that $\rho(b_1)/\rho_0$ is further reduced, to 0.28 and 0.18 for carbon and lead, respectively. Correspondingly, b_1 should be increased by about $0.1\ a = 0.06$ fm.

At $k_0/\mu = 1.25$, the density $\rho(b_1) \approx \frac{1}{2}\rho_0$ so that $x = 0$ in Eq. (8.8). Then Eq. (8.9) gives $\nabla^2\rho < 0$. However, the terms of higher order in ρ will now come in, and their effect, according to Krell and Ericson,[9] is likely to give another positive term in the parentheses of Eq. (8.7). We therefore believe that the correction to the LDA in this case is small.

X. WIDTH AND POSITION OF RESONANCE

The width of the resonance might be defined as the point where

$$\text{Im } k = \tfrac{1}{2}\ (\text{Im } k)_{max} \quad . \tag{10.1}$$

At the density $\rho = \frac{1}{2}\rho_0$, this occurs at a laboratory energy of 80 MeV (Table II). The nucleon-pion center-of-mass energy is then E = 1145 MeV while the resonance is observed [see Eq. (10.3)] to occur at E = 1220 MeV. The half-width is then

$$\tfrac{1}{2}\ \Gamma_{eff} = 65 \text{ MeV} \quad . \tag{10.2}$$

For scattering by a free proton, we may define the half-width as the point where $\delta_{33} = 45°$, which occurs at E = 1192 MeV while the resonance is at 1236, therefore

$$\tfrac{1}{2}\ \Gamma_{free} = 44 \text{ MeV} \quad . \tag{10.2a}$$

Thus the width in nuclear matter is considerably larger than for free nucleons. At full density ρ_0, it is still somewhat larger.

This result is in contrast to that of Dover and Lemmer[15] who find a <u>decrease</u> of the resonance width. This may be attributed to their assumption that $k = k_0$, which was discussed in Sec. I.

The width as defined by Eq. (10.1) is not directly observable. The Im k merely determines the transparency of the nucleus and thereby b_1 (Table II). Observations of the total (or reaction) cross section[64,65] show a very slow variation with energy impressively slower[65] than the π^+-proton cross section. These cross sections merely indicate the slow variation of b_1 with Im k.

Nevertheless, the observed reaction cross section may be used to get at least an estimate of the resonance width. The observations[64] seem compatible with our estimate [Eq. (10.2)], but more accurate evaluation will be necessary.

Much has been written about the position of the resonance. In our opinion, very little significance is attached to the energy at which the reaction[66] cross section reaches a maximum. This is only a pale reflection of the behavior of Im k, still further complicated by the finite wave length of the pion.

The only significant definition of the resonance is the energy where the real part of the nuclear scattered amplitude vanishes. The theoretical significance of this point was realized by Dedonder.[23] Experimentally, the real part of the amplitude may be measured by means of the Coulomb interference. This has been done for π^- by Binon et al.,[67] and for π^+ by Scott et al.,[68] in both cases using a ^{12}C target at laboratory energies from 115 to 260 MeV. Using either of these experiments, or better by comparing them, Re $f_N(0)$ can be determined. Interpolating between the data of Scott et al., we find the resonance at

$$T_{res} = 174 \ {}^{+21}_{-14} \text{ MeV} \qquad (10.3)$$

in the laboratory system. This compares with the (laboratory) energy of the resonance for scattering by the free nucleon,[41]

$$T_{free} = 195 \text{ MeV} \quad . \qquad (10.4)$$

Thus the experimental resonance position is shifted by

$$\Delta T_{exp} = -21 \ {}^{+21}_{-14} \text{ MeV} \quad . \qquad (10.5)$$

We previously (Sec. V, beginning) emphasized that the laboratory energy in a complex nucleus must be compared with the laboratory energy for free-nucleon scattering, not with the nucleon-pion center-of-mass energy.[69]

Our theory, Sec. VII, gave a shift

$$\Delta T_{the} = - 10 \text{ MeV} \quad . \qquad (10.6)$$

This is not sufficient, although it is within the wide limits of the experimental date [Eq. (10.5)]. Thus, the theory still requires modification. However, it has the great advantage over previous theories that our shift is composed of two very modest shifts, (a) from the propagation in nuclear matter, and (b) from the small phases. In other theories, many and individually large shifts are involved.

In Sec. VIII we emphasized that elastic scattering near the resonance gives no information about the behavior of pions in the high-density core of the nucleus because the nucleus is black. To investigate this behavior, we have to go off resonance, let us say to kinetic energies of 50 to 80 MeV. Silbar and Sternheim[12] have shown (their Fig. 2) that in this energy range, the "local" form [Eq. (8.7)] of the potential is in good agreement with the observed reaction cross section, whereas the original, nonlocal form [Eq. (8.1)] of the Kisslinger theory gives too large reaction cross sections. We believe this may be explained by the fact that the local potential [Eq. (8.7)] does not suffer from the "Kisslinger syndrome" [Eq. (1.6)] at large ρ, while it is similar to Kisslinger's in the nuclear surface. As we have discussed in this report, a correct theory should indeed have these two properties. (Of course, this does not justify the details of the local theory in the nuclear interior.)

XI. ELASTIC SCATTERING AND ^{48}CALCIUM

In Sec. IX, we used the eikonal method to estimate the elastic scattering. Using Eqs. (9.9), (9.4), and (9.12), the source of elastic scattering is given by

$$S(b) = s(b)e^{-ik_0 z_1} = \exp X(b) - 1 \quad . \qquad (11.1)$$

Following the usual methods, the amplitude of the diffracted wave at large distances becomes

$$2\pi k F(\theta) = - 2\pi i k \int b db S(b) J_0(k_0 b \sin\theta) \qquad (11.2)$$

Probably better accuracy will result if we replace

$$\sin\theta \to 2 \sin\theta/2 \quad . \qquad (11.2a)$$

According to Sec. IX, if the center of the nucleus is black, we have approximately

$$S(b) = - 1 \qquad \text{for } b < b_1 \quad . \qquad (11.3)$$

For $b > b_1$, Eq. (9.12) gives

$$X(b) = i\,\phi(\omega)\,(2\pi ab)^{\frac{1}{2}}\,\rho(b)/\rho_0 \qquad (11.4)$$

where we have added the true-absorption term $C\rho_0$ to Eq. (9.14). Now use the radial distribution [Eq. (9.10)],

$$\rho(b) = \rho(b_1)\,\exp(b_1 - b)/a \quad . \qquad (11.5)$$

Then Eq. (11.2) becomes

$$F(\theta) = -i\int_0^{b_1} bdb(-1)\,J_0(qb) - i\int_{b_1}^{\infty} bdb\,J_0(qb)\left\{\exp\left[X(b_1)e^{(b_1-b)/a}\right]-1\right\} \qquad (11.6)$$

with

$$q = 2k_0 \sin\theta/2 \quad . \qquad (11.6a)$$

The first integral is elementary and gives the well-known diffraction from a black disk.

$$F_1(\theta) = ib_1J_1(qb_1)/q \quad . \qquad (11.7)$$

The second integral may be simplified by noting that $a \ll b_1$ so that only values of b close to b_1 contribute. Than b can be replaced by b_1 everywhere except in the exponential and in db, thus

$$F_2(\theta) = -ib_1aJ_0(qb_1)\int_0^{X(b_1)} \frac{dX(b)}{X(b)} \times [\exp X(b) - 1] \qquad (11.8)$$

It should be remembered that X has a negative real part [Eq. (9.15)]. Then

$$F_2(\theta) = ib_1aJ_0(qb_1)\left\{\ln\left[-X(b_1)\right] + C - Ei\ X(b_1)\right\} \quad , \qquad (11.9)$$

where $C = 0.577$ is Euler's constant. The integral-exponential $Ei\ (X(b_1))$ is small, and it is essentially cancelled by the exponential term which we left out in the first integral in Eq. (11.6). Using Eqs. (9.15), (11.4), and (9.11), we have

$$-X(b_1) = \ln 2\left[1 - i\frac{\mathrm{Re}\ \phi}{\mathrm{Im}\ \phi + C\,\rho_0}\right] = \ln 2\left[1 - i\frac{\mathrm{Re}\ k(\rho,\omega) - k_0}{\mathrm{Im}\ k(\rho,\omega) + C\rho}\right] \quad , \qquad (11.10)$$

where $k(\rho,\omega)$ is evaluated at a suitable value of ρ near $\rho(b_1)$. Then setting

$$Y = \frac{\mathrm{Re}\ k(\rho,\omega) - k_0}{\mathrm{Im}\ k(\rho,\omega) + C\rho} \qquad (11.11)$$

and inserting into Eq. (11.9), we get

$$F_2(\theta) = ib_1aJ_0(qb_1)\left[C + \ln(\ln 2) + \tfrac{1}{2}\ln(1 + Y^2)\right] + b_1J_0(qb_1)\ \arctan Y \quad . \qquad (11.12)$$

Adding this to Eq. (11.7) gives the total scattered amplitude.

The imaginary part of the scattered amplitude is mainly the contribution from the black disk [Eq. (11.7)]. The first term in Eq. (11.12) is small. The second term gives the real scattering amplitude; it has the same sign as Eq. (11.11), hence it is positive if the potential on the pion is attractive. The real part of Eq. (11.12) may permit at least semiquantitative comparison with Coulomb interference at energies away from the resonance. The real part arises only from impact parameters near b_1, as shown by the factor $J_0(qb_1)$.

Diffraction minima correspond to a zero of the imaginary part of F, therefore essentially to

$$J_1(qb_1) = 0 \quad . \qquad (11.13)$$

The differential cross section for this q should be zero at the resonance energy. Away from resonance, the real part of Eq. (11.12) will still give a non-vanishing result. This may be another way to ascertain the position of the resonance.

It has often been suggested that pions in the resonance region would be a very good tool for the exploration of the neutron distribution in such nuclei as ^{48}Ca. The proton distribution in this nucleus is known to be very nearly the same as in

^{40}Ca, but in addition there is a complete shell of neutrons containing eight of these in the 1 $f_{7/2}$ shell. Figure 13, curve C, shows the density distribution of the neutrons or protons in the inner shells (1s 1p 1d 2s) of ^{48}Ca, according to the density-dependent Hartree-Fock calculations of Negele[70] which agree well with electron scattering experiments and measured binding energies. Curve f shows the density of the 1 f neutrons on the same scale. Beyond r = 5 fm, the 8 f neutrons contribute somewhat more to the total density than the 20 core neutrons.

The scattering of π^- is determined by the effective density

$$\rho\ (\pi^-) = \tfrac{3}{2}\,\rho_n + \tfrac{1}{2}\,\rho_p \quad , \tag{11.14}$$

where ρ_n and ρ_p are the neutron and proton density. The effective density for π^+ scattering is

$$\rho\ (\pi^+) = \tfrac{1}{2}\,\rho_n + \tfrac{3}{2}\,\rho_p \quad . \tag{11.15}$$

Both $\rho(\pi^-)$ and $\rho(\pi^+)$ are shown on Fig. 13. We have marked with crosses the points where $\rho(\pi^+)$ and $\rho(\pi^-)$ become equal to $\frac{1}{4}\,\rho_0 = 0.040$ fm^{-3} which we showed in Table II to be about the point b_1, i.e., the effective nuclear radius for elastic scattering near the resonance energy. The corresponding point for core scattering alone is marked on the C curve.

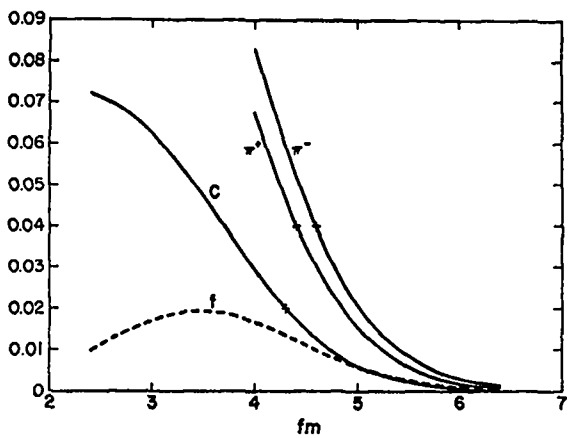

Fig. 13 Density distribution of nucleons in ^{48}Ca according to the calculations of Negele[70] in nucleons per fm^3. Curve C: Core nucleons (either neutrons or protons). Curve f: Neutrons in the 1 f shell. Other curves: Effective density for scattering of π^+ or π^-. The + signs mark the points where the effective density is $1/4\rho_0 = 0.04$ fm^{-3}.

The effective radii are

$$\begin{aligned} b_1\ (\pi^-) &= 4.60 \text{ fm}\ , \\ b_1\ (\pi^+) &= 4.41 \text{ fm}\ , \text{ and} \\ b_1\ (C) &= 4.30 \text{ fm}\ . \end{aligned} \tag{11.16}$$

Thus the ^{48}Ca nucleus should appear about 4% larger for π^- than for π^+ scattering, and the latter radius is about 3% larger[71] than the radius for scattering by ^{40}Ca. The 4% difference between π^+ and π^- should be easy to observe by the position of the first diffraction minimum; this would give direct evidence that the neutron f shell extends beyond the core.

There is some chance that a similar, but much smaller, effect could be observed in heavy nuclei like ^{208}Pb. Whereas the maximum of the neutron shell is located at about the same radius as the last proton shell,[70] the proton density decreases more rapidly for large r than the neutron density because of the Coulomb potential. Aside from the effect being smaller, however, the Coulomb scattering might mask the first diffraction minimum in a nucleus as heavy as lead.

Another way to observe the neutron 1 f shell in ^{48}Ca may be the charge exchange scattering. We consider in particular the process in which the ^{48}Ca nucleus is changed into the analog state in ^{48}Sc,

$$\text{Ca} + \pi^+ = {}^{48}\text{Sc}(T = 4) + \pi^\circ \quad . \tag{11.17}$$

The analog state has isospin T = 4, and a wave function just like ^{48}Ca, except that one neutron in the 1 f shell is replaced by a proton. The energy change is just that due to the Coulomb energy, about 6 MeV. The process may be recognized by selecting π°'s of high energy corresponding to the 6-MeV energy loss.[72] The process (11.17) can only go on with the neutrons in the 1 f shell. The core neutrons will also give charge exchange, but this involves much greater energy transfer to the nucleus. Moreover, the process (11.17) can go on with π^+; the π^- can only undergo charge exchange with core protons. So the observation of the difference between $\pi^- - \pi^\circ$ and $\pi^+ - \pi^\circ$ charge exchange is another way to separate out the process (11.17). Once this is done, it should give the density distribution of the 1 f neutrons, rather than the combinations in Eq. (11.14)

and Eq. (11.15); but of course the absorption of the π^+ before charge exchange, and of the π° afterwards, must still be taken into account, and these involve Eq. (11.15) and $\rho(\pi^\circ) = \rho_n + \rho_p$.

Returning to the general problem of elastic scattering, a correct theory should of course not use the eikonal method, but an analysis in angular momenta. This has been attempted by Beiner and Huguenin[73] who write scattered amplitude[74]

$$f(k_0, \theta) = (2ik_0)^{-1} \sum_{\ell=0}^{L} (2\ell+1)\ (S_\ell - 1)\ P_\ell\ (\cos\theta) \tag{11.18}$$

where

$$S_\ell = e^{2i\delta_\ell} \tag{11.18a}$$

with complex phase shift δ_ℓ. When doing such an analysis, one should be guided by the physical features of the scattering as discussed in Secs. IX and XI. In the resonance region, let us say from 100 to 250 MeV (laboratory kinetic energy), the nucleus is black, so that we have very nearly

$$S_\ell = 0 \quad \text{for } \ell \leq \ell_c = k_0 R - \tfrac{1}{2} \ . \tag{11.19}$$

For ^{12}C, R = 2.58 fm, we get

$$\ell_0 = 1.75 \text{ for } T_{lab} = 100 \text{ MeV}$$
and
$$\ell_0 = 3.5 \text{ for } T_{lab} = 250 \text{ MeV} \ . \tag{11.20}$$

Only for $\ell > \ell_0$ do we need to find S_ℓ from the experimental data. The maximum useful L may be set arbitrarily by $S_L = 0.9$. Using the notation of Sec. IX, this corresponds to

$$b_2 = b_1 + 1.86\ a \tag{11.21}$$
and
$$L = k_0 b_2 - \tfrac{1}{2} \ . \tag{11.22}$$

For ^{12}C, we have $b_2 = 4.35$ fm,

$$L = 3.3 \text{ for } T_{lab} = 100 \text{ MeV}$$
and
$$L = 6.2 \text{ for } T_{lab} = 250 \text{ MeV} \ . \tag{11.23}$$

Thus we need only consider[75]

$$\ell = 2 \text{ and } 3 \text{ for } 100 \text{ MeV}$$
and
$$\ell = 4,\ 5 \text{ and } 6 \text{ for } 250 \text{ MeV} \ . \tag{11.24}$$

It should be possible to do a phase shift analysis, even with complex phases, for such a small number of partial waves. This will be a way to <u>derive</u> $k(\rho,\omega)$ <u>from experiment</u>, rather than predict the experimental results from an imperfect theory.

XII. INELASTIC SCATTERING AND REFRACTIVE INDEX

We have seen in Secs. IX-XI that the elastic scattering in the resonance region does not indicate anything about the behavior of pions in the <u>interior</u> of the nucleus, except that the imaginary part of k is large. We shall now examine what information we can get from inelastic scattering. In particular, we shall look at quasi-elastic scattering, i.e., the scattering of a pion by a single nucleon (see also Sec. V.

The main feature of the theory is that the <u>real</u> part of k in the nucleus is much greater than k_0, provided the energy is appreciably below resonance (175 MeV lab). Accordingly, the energy transfer to the nucleon is also much larger. From a quasi-elastic scattering, we therefore expect to get
(1) a recoil nucleon of very high energy, and
(2) an inelastic pion of low energy.
As we show below, it is probably difficult to observe the pion, so we shall concentrate on the recoil nucleon.

Quasi-elastic scatterings may occur (a) in the nuclear surface, let us say at $r > R$, where R is the point where $\rho = \tfrac{1}{2}\rho_0$, and (b) in the interior, $r < R$. In the surface region, the pion behaves almost like a free pion, the nucleon effective mass is near 1, and the nucleon momentum is not very large because $p_F < 1.05$ fm^{-1}. Assuming the nucleon to be initially at rest and the pion to be scattered backward, the energy transfer is

$$\Delta\omega = \frac{p_F^2}{2M} = \frac{2\ k_0^2 M}{(M + \omega_1)^2} \ . \tag{12.1}$$

This is roughly <u>half</u> of the kinetic energy of the pion. The nucleon recoil energy is about $\Delta\omega - B$ where $B \approx 8$-16 MeV is the binding energy.

In the interior, the momentum of the pion is much larger than k_0 which permits a larger energy transfer. The larger nucleon momentum, and the smaller M*, further enhance the energy transfer. As we discussed in Sec. V, nearly all of the pion kinetic energy may be transferred to the nucleon. Thus we may expect to see nucleons of high energy emerge from the interior of the nucleus.

It is true that these nucleons may suffer collisions while trying to get out of the nucleus. In these collisions they may lose energy, be deflected, or suffer charge exchange. But the collision cross section is only about 40 mb, roughly half that of a resonance pion, so that a recoil nucleon has a much better chance to emerge, possibly even with undiminished energy and with its original charge. We therefore propose an experimental search for high-energy, quasi-elastic recoil nucleons. The incident pion energy should be well below the resonance, perhaps 100 MeV.

The probability that a resonance pion, hitting the nucleus head-on, will penetrate to $\rho = \frac{1}{2}\rho_0$, $r = R$ is

$$P = \exp\left(-2\int_R^\infty \mathrm{Im}\, k(r)\, dr\right) . \qquad (12.2)$$

Using Fig. 10, Eqs. (9.10) and (9.11), and Table II, we find

$$P \approx 0.6 . \qquad (12.3)$$

For off-center hit, P is smaller, but we estimate that roughly half of the pions have their first inelastic collision in the interior.

The scattered pions themselves have considerable difficulty in getting out of the nucleus. In the first place, there is the absorption by Im k, giving a second factor P. More interesting, and often more important, is the refractive index

$$n = k/k_0 . \qquad (12.4)$$

Using Fig. 11, we have calculated Re n as a function of ω, at $\rho = \rho_0$. Figure 14 shows the result: Re n has a maximum of nearly 2 between 30 and 60 MeV, then falls to 1.5 at 100 MeV, 1.2 at 150 MeV, and to 1.0 at 220 MeV (lab). The minimum is about 0.94 at 300 MeV.

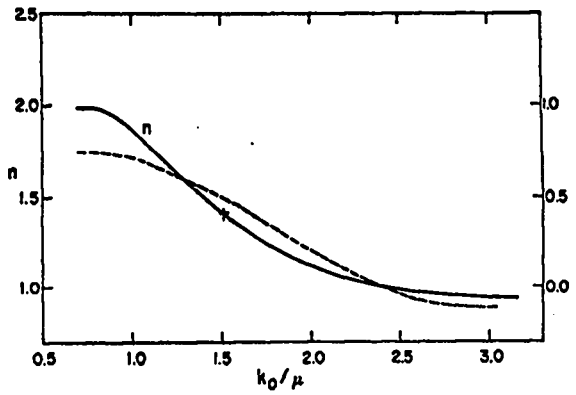

Fig. 14 Real part of the refractive index, at density ρ_0, as a function of k_0/μ. The + sign marks $n = \sqrt{2}$ where the probability of total reflection is one-half. ---line of $\cos^2\theta_c$, read on the right-hand scale.

A pion of 30 to 150 MeV, coming from the inside to the nuclear surface (now considered plane) may thus suffer total reflection. This phenomenon, as is well-known, is not modified by the fact that n changes gradually from its interior value to 1. Total reflection will occur if the angle θ between the pion momentum and the normal to the surface satisfies Snell's condition

$$\sin\theta > \sin\theta_c = 1/n . \qquad (12.5)$$

Assuming that the direction of the pions in the interior (e.g., after quasi-elastic scattering) is random, the fraction unable to get out is

$$\cos\theta_c = (1 - 1/n^2)^{\frac{1}{2}} . \qquad (12.6)$$

Actually, because of the absorption of pions in the interior, both on the way in and especially on the way out, the direction of pions coming to the surface is not random, but favors small angles θ. An uninteresting calculation shows that the fraction totally reflected is then not Eq. (12.6) but nearly

$$\cos^2\theta_c = 1 - 1/n^2 . \qquad (12.7)$$

This quantity is also plotted in Fig. 14. Up to about 110 MeV, the chance of getting totally reflected is greater than one-half!

It is for this reason that the observation of quasi-elastically scattered pions is apt to be difficult. Their number is small, and the theory contains so many uncertain factors that interpretation of the cross section is nearly impossible. Even if inelastic pions are observed, it is hard to tell whether they have been scattered once or several times. An exception is, probably, the case of very low energy of the scattered pion; see below.

The large total reflection does, however, have one simple consequence. Most pions, once they are inelastically scattered, will stay in the nucleus for a long time, and therefore are apt to be captured by the two-nucleon process [Eq. (1.11)]. This process by itself has a very small cross section;[62] even at $\rho = \rho_0$, the corresponding Im k is only about 0.035 μ, corresponding to a mean free path of 20 fm. However, by repeated total reflections, the path of the pion inside the nucleus may indeed be 20 fm or more. The capture process [Eq. (1.11)] should be observable because the two nucleons resulting from it have high energy because they have absorbed the rest energy μc^2 of the pion. But they may not come out in opposite directions because the pion momentum is still appreciable, perhaps as high as 2 μ. Their charge is also not necessarily predictable because the pion may have suffered one or several charge exchanges.

An integral method to determine this capture of fast pions is to send a π^+ beam on a thick target of some material, and to measure the positrons from the $\pi^+ - \mu^+ - e^+$ decay. These positrons will be emitted when the π^+ has been brought to rest. Their number, divided by the number of incident π^+, indicates how many π^+ escape capture in flight.

The total reflection, which we have discussed, may stop at low pion energy. As we discussed in Sec. VI, Re k may become small and even negative at small pion energy. Thus at some energy ω_1 (perhaps 30 MeV), Re k = k, and below this energy,

$$\text{Re } n < 1 \quad \text{if } \omega < \omega_1 \quad . \tag{12.8}$$

For these low energies, then, the pion will come out easily from the nucleus once it has made its way to the surface. Thus we may expect that many of the inelastically scattered pions will come out with very low energy regardless of the energy of the incident pion.[76] The energy distribution of these pions will give information on the optical potential of pions in the low-energy region, which is very interesting. But it will give no information on the inelastic scattering processes which have taken place in the interior because most of the pions will have suffered more than one collision. The angular distribution of these low-energy pions is probably not far from isotropic, with some preference for the backward direction (because many of them will come out near the place where the incident pion has entered).

The result, [Eq. (12.8)], may also mean that incident pions of low energy, $\omega < \omega_1$, may experience something like mirror reflection from the nucleus, giving enhanced and nearly isotropic elastic scattering.

At low energy, the term $\nabla^2\rho$ in Eq. (8.7) becomes important (i.e., the local density approximation is no longer good). Then there is effectively a potential minimum at the nuclear surface, and the pions may preferentially move along the surface.

We have discussed only one type of inelastic scattering, the quasi-elastic one. There are at least two other types: (a) excitation of surface vibrations of the nucleus, and (b) ejection of α-particles. Both of these may happen particularly with pions moving near the surface of the nucleus. In case (b) this is so because only in this manner can the α-particle get out easily.

Experiments by Jackson et al.,[77] by Lind et al.,[78] and by Ashery et al.,[79] indicate that emission of one or several α-particles is remarkably frequent. The experiment consists in the observation of the characteristic γ rays from the residual nucleus which indicate how many nucleons and of what charge have been removed from the target nucleus. The incident pions had energies up to 380 MeV. Experiments by Lewis et al.,[80] at 60 and 0 MeV, indicate a considerably smaller effect of this kind.

H. Lipkin[81] has suggested that the present theory may help explain these observations. First, the pions are often near the nuclear surface, as discussed above. Second, they may not have enough energy to eject single nucleons. In collisions with α-particles (perhaps preformed in the nuclear surface) the pion loses less energy because of the greater mass of the alpha.

It is obvious that the behavior of low-energy pions is far from clear, and will require a lot of further theoretical and experimental work.

XIII. SUMMARY

The basic quantity in our theory is the pion self-energy Π which gives the relation between the wave number $k(\rho,\omega)$ of a pion in the nucleus and the energy ω [Eq. (2.2)]. Following the work of Dover and Lemmer,[15] the main part of Π is obtainable from the amplitude for forward scattering of a pion by a single nucleon in nuclear matter by integration over the nuclear density [Eq. (2.11)]. The main modification to this result is the Lorenz-Lorentz correction [Eq. (2.12)] which is due to the fact that two nucleons cannot come very close together.[19]

The forward scattering amplitude $f(\omega,k,\rho)$ is obtained from a theory analagous to that of Chew and Low.[16] The most important problem here is unitarity. To treat this problem properly, we have to go to real values of k, hence complex values of ω. This introduces certain modifications in the Chew-Low theory. However, it is shown that the main feature of the Chew-Low theory, the "effective range approximation," is preserved, and we get a 33-resonance at about the same energy as for free nucleons. After performing the calculations, we can transform back to real energy ω and complex k, and obtain the final result for forward scattering [Eq. (3.31)].

The Pauli principle for the nucleons in nuclear matter is important because it excludes certain final states in the quasi-elastic scattering of the pion by a single nucleon. It is taken into account by solving the Lippman-Schwinger equation (Sec. IX and App. A). It is helpful in this connection that f is only needed for nucleons at the top of the Fermi sea.

The (real part of the) pion momentum in nuclear matter is large. This leads to large energy loss of the pion in quasi-elastic collisions once the finite mass of the nucleon is taken into account. Like the Pauli principle, this reduces the probability of quasi-elastic scattering, and hence the imaginary "damping" term which this process introduces into the elastic scattering amplitude. The combined effect is about a factor of 1.28 in this damping (Sec. V and App. B). Also the form of the damping term is changed. In this respect, our theory differs from the simple model[19] in which the pion changes the nucleon temporarily into a Δ; in other respects, this model is most similar to our theory.

The behavior of pions is rather different at low energy where the damping by inelastic scattering is small. The standard theory[5,9,10] of pionic atoms is found justified. At kinetic energies up to about 30 MeV, $k(\rho,\omega)$ is calculated algebraically (Sec. VI); it is mostly imaginary.

At energies above 30 MeV, a simple model of our theory is calculated numerically (Sec. VII). The wave number k remains moderate for all densities, so that the Kisslinger[1] syndrome [Eq. (1.6)] is avoided. The real part of k is always much larger than the imaginary part, again in contrast to the Kisslinger theory. The effective absorption cross section at resonance is considerably less than the free-nucleon-pion cross section.

Effects near the nuclear surface are considered in a very approximate way (Sec. VIII). Then the scattering by a finite nucleus is treated in the eikonal approximation (Sec. IX). The elastic scattering of resonance pions is essentially determined by the part of the nucleus where the density is about a quarter of nuclear density. Because at this low density the Kisslinger (and similar) theories are valid, this explains the fact that these theories have given good numerical answers for elastic scattering. A reason is given why the "local" form of that theory [Eq. (8.7)] may be especially successful.

The center of the nucleus is black to resonance pions. To get information on the interaction of pions with high-density nuclear matter, we must use pions of lower energy, perhaps 30-80 MeV. Our theory is not yet ready to describe these because we have not yet dealt properly with the effects of changing density (surface effects).

The width of the resonance in our theory is appreciably greater than for free nucleons (Sec. X), in contrast to some recent theories.

The position of the resonance should not be defined by the maximum of the reaction cross section, and even less of the total cross section, because these are secondary effects. The only proper definition is as the energy at which the real part of the nuclear scattered amplitude is zero. This can

be experimentally measured by the Coulomb interference. Experiments show that the so-defined resonance is shifted downward from the free-nucleon resonance by 20 ± 20 MeV. Our theory predicts a downward shift by 10 MeV, which is the sum of a slight upward shift according to the calculations of Sec. VII, plus a somewhat larger down shift due to the contribution of the "small" waves in pion-nucleon scattering. We do not consider this agreement as satisfactory.

The eikonal method is used for a simple quantitative calculation of elastic scattering (Sec. XI). For a direct analysis of experimental data, one may analyze for the complex phase shifts of those partial waves which correspond to impact parameters near the nuclear radius; only 2 or 3 values of ℓ will be needed. For the small values of ℓ. the nucleus may be considered black (Sec. XI).

The proposed experiment to determine the "neutron halo" of ^{48}Ca is very promising. The apparent nuclear radius for scattering of π^- is about 4% larger than for π^+ (Fig. 13). Charge-exchange scattering by ^{48}Ca is also promising.

A conspicuous feature of pion physics in nuclear matter is the large refractive index, Re $n \approx 2$ at about 50 MeV. Pions of energies between about 30 and 120 MeV which try to emerge from the interior of the nucleons are apt to suffer total reflection. Therefore there should be few inelastically scattered pions in this energy range. At low energies, < 30 MeV, inelastically scattered pions may escape from the nucleus. In many cases, however, inelastic scattering in the interior will lead to subsequent capture of the pion in the same nucleus. Some experiments are proposed to study these phenomena (Sec. XII).

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APPENDIX A

PAULI DAMPING FOR $P_{3/2}$ SCATTERING

For a $p_{3/2}$ state, the K matrix element depends on the initial and final directions not only of the momentum, but also of the spin, thus,[44]

$$\langle \vec{k}' \; m' | K | \vec{k} \; m \rangle = - 8\pi^2 K_0 \times \tag{A.1}$$

$$\times \sum_M \langle m' | y_{3/2 \; M}(\hat{k}') \rangle \langle y_{3/2 \; M}(\hat{k}) | m \rangle \quad ,$$

where y_{jM} is the generalized spherical harmonic. Equation (A.1) suggests that we write the T matrix in the form

$$\langle \vec{k}' m' | T | \vec{k} \; m \rangle = \sum_M \langle m' | y_{3/2 \; M}(\hat{k}') \rangle \tag{A.2}$$

$$\langle \vec{k}' \; M | T | \vec{k} \; m \rangle \quad .$$

Equation (4.5) is modified merely by adding the spin variables m_i, m, and m_f in the matrix elements. Inserting Eq. (A.1,2), we get

$$\sum_M \langle m_f | y_{3/2 \; M}(\hat{k}_f) \rangle \langle k \; M | T | \vec{k}_i \; m_i \rangle$$

$$= -8\pi^2 K_0 \sum_M \langle m_f | y_{3/2 \; M}(\hat{k}_f) \rangle \langle y_{3/2 \; M}(\hat{k}_i) | m_i \rangle \tag{A.3}$$

$$+ ik\omega \, K_0 \sum_m \int_{-1}^{x} d(\cos\chi) \int d\psi \sum_M \langle m_f | y_{3/2 \; M}(\hat{k}_f) \rangle$$

$$\langle y_{3/2 \; M}(\hat{k}) | m \rangle \sum_{M'} \langle m | y_{3/2 \; M'}(\hat{k}) \rangle \langle k \; M' | T | \vec{k}_i \; m_i \rangle \; .$$

Clearly, we can pick out the coefficient of

$$\langle m_f | y_{3/2 \; M}(\hat{k}_f) \rangle$$

on both sides, and thereby reduce the sum over M to one term. Next we calculate the integral over ψ and sum over m. We note that

$$y_{3/2 \; M}(\hat{k}) = \sum_n C(1n, \tfrac{1}{2} \; M\text{-}n; \, 3/2 \; M) \; Y_1^M(\hat{k}) \, | M\text{-}n \rangle, \tag{A.4}$$

where the C's are Clebsch-Gordan coefficients, the Y's are ordinary spherical harmonics, and $|M\text{-}n\rangle$ is the spin function of $m_s = M\text{-}n$. Then

$$I_M = \int d\psi \sum_m \langle y_{3/2 \; M}(\hat{k}) | m \rangle \langle m | y_{3/2 \; M'}(\hat{k}) \rangle$$

$$= \sum_m C(1 \; M\text{-}m, \tfrac{1}{2}m; \, 3/2 \; M) \; C(1 \; M'\text{-}m, \tfrac{1}{2}m; \, 3/2 \; M')$$

$$\int d\psi \; Y_1^{M-m}(\hat{k})^* \; Y_1^{M'-m}(\hat{k}) \quad . \tag{A.5}$$

The integral over ψ is zero unless $M' = M$. Then

$$I_M = \sum_n C^2(1 \; n, \tfrac{1}{2} \; M\text{-}n; \, 3/2 \; M) \; 2\pi |Y_1^n(\hat{k})|^2 \quad , \tag{A.6}$$

$$2\pi |Y_1^1(\hat{k})|^2 = 3/4 \sin^2\chi \quad , \tag{A.7}$$

$$2\pi |Y_1^0(\hat{k})|^2 = 3/2 \cos^2\chi \quad , \tag{A.7b}$$

$$C^2(1 \; M\text{-}\tfrac{1}{2}, \tfrac{1}{2} \; \tfrac{1}{2}, \; 3/2 \; M) = \tfrac{1}{2} + M/3 \; , \text{ and} \tag{A.8a}$$

$$C^2(1 \; M\text{+}\tfrac{1}{2}, \tfrac{1}{2} \; \text{-}\tfrac{1}{2}, \; 3/2 \; M) = \tfrac{1}{2} - M/3 \quad . \tag{A.8b}$$

Because $M' = M$, Eq. (A.3) reduces to an equation for the single unknown $\langle k \; M | T | \vec{k}_i m_i \rangle$, which is

$$\langle k \; M | T | \vec{k}_i m_i \rangle = - 8\pi^2 K_0 \langle y_{3/2 \; M}(\hat{k}_i) | m \rangle \tag{A.9}$$

$$+ ik\omega K_0 \langle k \; M | T | \vec{k}_i m_i \rangle \int_{-1}^{\lambda} d(\cos\chi) \; I_M(\chi)$$

and is easily solved for T. According to Eq. (A.7), we clearly need the integrals [see also Eq. (4.20].

$$J_C = 3/2 \int_{-1}^{\lambda} \cos^2\chi \; d(\cos\chi) = \tfrac{1}{2}(1 + \lambda^3) \tag{A.10a}$$

$$J_S = 3/4 \int_{-1}^{\lambda} \sin^2\chi \; d(\cos\chi) = \tfrac{1}{4}(1 + \lambda)(2 + \lambda - \lambda^2) \; . \tag{A.10b}$$

Using Eq. (A.8),

$$J_M = \int_{-1}^{\lambda} d(\cos\chi) \; I_M(\chi) = J_s \quad \text{if} \quad M = \pm 3/2 \tag{A.11a}$$

and

$$J_M = 2/3 \; J_C + 1/3 \; J_s \qquad \text{if} \quad M = \pm \tfrac{1}{2} \quad . \tag{A.11b}$$

Then Eq. (A.9) has the solution

$$\langle kM|T|\vec{k}_i m_i\rangle = -8\pi^2 K_o \langle y_{3/2\ M}(\hat{k}_i)|m_i\rangle\, t_M \qquad (A.12)$$

and

$$t_M = (1 - ik\omega K_o J_M)^{-1} \quad . \qquad (A.12a)$$

Inserting in Eq. (A.2),

$$\langle \vec{k}_f m_f|T|\vec{k}_i m_i\rangle = -8\pi^2 K_o \sum_M t_M \times$$

$$\times \langle m_f|y_{3/2\ M}(\hat{k}_f)\rangle \langle y_{3/2\ M}(\hat{k}_i)|m_i\rangle \quad . \qquad (A.13)$$

We are chiefly interested in the forward scattering, $\hat{k}_f = \hat{k}_i$. Define

$$F_M = 8\pi \langle m_i|y_{3/2\ M}(\hat{k}_i)\rangle \langle y_{3/2\ M}(\hat{k}_i)|m_i\rangle + \text{same with } -M \quad . \qquad (A.14)$$

Recalling Eqs. (A.4) and (A.8), we find for $m_i = m_f = \frac{1}{2}$

$$F_{3/2} = 8\pi|Y_1^{\ 1}|^2 = 3\sin^2\chi_i \qquad (A.15a)$$

and

$$F_{\frac{1}{2}} = 8\pi\left[{}^2/_3|Y_1^{\ 0}|^2 + {}^1/_3|Y_1^{\ 1}|^2\right] = 3\cos^2\chi_i + 1 \quad . \qquad (A.15b)$$

For $m_i = -\frac{1}{2}$, the same results are obtained. For $m_i = \frac{1}{2}$, $m_f = -\frac{1}{2}$ (spin-flip), it is easy to show that the contributions of $M = +\frac{1}{2}$ and $-\frac{1}{2}$ to Eq. (A.13) just cancel; they have the same t_M. Thus we find

$$\langle \vec{k}_i m'|T|\vec{k}_i m\rangle = -\pi\delta_{mm'} K_o \sum_M t_M F_M \qquad (A.16)$$

where the sum goes only over the <u>two</u> values M = 3/2, 1/2, and F_M is given in Eq. (A.15).

APPENDIX B
CALCULATION OF A MODEL

The assumptions of the model are stated in Eqs. (5.23) and (5.24). We first calculate the angle χ_f of a pion which ends up at $\omega_f = \omega_c$ and $k_f = k_i = k_r$. Remembering Eq. (4.11a), Eq. (5.7) gives

$$\cos\chi_i - \cos\chi_f = M^*(\omega_i - \omega_c)/Pk_r \quad . \qquad (B.1)$$

From our estimate in Eq. (5.22), it is clear that nearly always

$$\cos\chi_f > -1 \quad . \qquad (B.1a)$$

The only exception occurs if P is very small, $\cos\alpha \approx -1$ in Eq. (5.22), i.e., $\vec{p}_i$ nearly opposite to $\vec{k}$. We shall come back to this case.

In the usual case, i.e., if Eq. (B.1a) is true, we have to consider two ranges of integration, viz.,

a. from χ_i to χ_f; here k is constant = k_i and ω changes with χ, from ω_i to ω_c.
b. from χ_f to π; here $\omega = \omega_c$ is fixed, and k changes with χ, from k_i to k_m (minimum k) which we shall calculate.

In range a, we get ω from Eq. (5.7) which we write in the form

$$\omega = \omega_2 + (P/M^*)\, k\cos\chi \qquad (B.2)$$

with

$$\omega_2 = \omega_i - (P^2 + k_i^{\ 2} - p_F^{\ 2})/2M^* \quad . \qquad (B.3)$$

If we assume $k_i = p_F$, as in Eq. (5.24), the last term in Eq. (B.3) is about $P^2/2M^*$. An average value of P^2 is $P_F^2 + k^2$, then using Eq. (5.26)

$$P^2/2M^* = 0.83\ \mu \quad . \tag{B.3a}$$

If ω_i is the resonance energy,

$$\omega_i = 2.4\ \mu,\ \omega_2 = 1.57\ \mu \quad . \tag{B.3b}$$

Further, in Eqs. (5.9) and (B.2), using Eq. (5.26) again,

$$\beta = P/(M + \omega_i) = 0.316 \quad , \tag{B.4}$$

$$P/M^* = C\beta \quad , \quad \text{and} \tag{B.4a}$$

$$C = (M + \omega_i)/M^* = 1.82 \quad . \tag{B.4b}$$

We insert Eq. (B.2) into Eq. (5.11), neglecting consistently all terms of order[49] β^2, in the following.

$$x = k' \cos\chi' = k\cos\chi - \omega_2\beta \tag{B.5}$$

We solve this for k cos χ and insert in Eqs. (B.2) and (5.14)

$$\omega = \omega_2 + \beta\, C\, x \quad , \quad \text{and} \tag{B.6}$$

$$\omega' = \omega_2 + \beta(C-1)x \quad . \tag{B.6a}$$

This shows that ω' is a function of the scattering angle, i.e., of x, as noted below Eq. (5.15). The reason is partly the small effective mass of the nucleon [see Eq. (B.4b)]. From Eq. (5.13),

$$k'^2 = k^2 - 2\beta\,\omega_2\, x \quad . \tag{B.7}$$

Therefore, as x changes (k = constant)

$$d(k'^2) = 2\beta\omega_2\, dx \quad . \tag{B.7a}$$

As we stated above Eq. (5.21), we use k'^2 instead of $k'^2\cos^2\chi'$ in the integral in Eq. (5.19). Omitting the factor μ^{-2} and using Eq. (5.20), we have from range a.

$$I = \int_{\chi_f'}^{\chi_i'} \omega'\, k'^3\, d(\cos\chi')$$

$$= \int_{x_2}^{x_1} [\omega_2 + \beta(C-1)x]\, \left[(k^2 - 2\beta\omega_2 x)dx - \tfrac{1}{2}x(-2\beta\omega_2\, dx)\right]$$

$$= k^2\,\omega_2\, x + \tfrac{1}{2}\beta\,[k^2(C-1) - \omega_2^2]\, x^2 \,\Big|_{x_1}^{x_2} \quad , \tag{B.8}$$

neglecting again a term β^2. The limits on x are as follows [see Eqs. (B.5), (5.7), and (B.1).

$$x_1 = k_i \cos\chi_i - \omega_2\beta$$
$$= \tfrac{1}{2}\,P - \omega_2\beta + (k^2 - p_F^2)/2P = 1.18\ \mu \quad , \tag{B.9}$$

$$x_1 - x_2 = (M^*/P)\,(\omega_i - \omega_c) = 1.51\,(\omega_i - \omega_c)$$
$$= 1.81\ \mu \quad , \tag{B.10}$$

$$x_2 = -\,0.63\ \mu \quad , \quad \text{and} \tag{B.10a}$$

$$x_1^2 - x_2^2 \text{ very small} \quad . \tag{B.10b}$$

The last term in Eq. (B.8) is therefore negligible; the bracket in this term is also rather small, and so is the factor $\tfrac{1}{2}\beta = 0.16$. In Eq. (B.10), we have used $\omega_c = 1.2\ \mu$ (kinetic energy 30 MeV). Thus we have essentially

$$I = k^2\,\omega_2\,(x_1 - x_2) =$$

$$= k_i^2\,\omega_2\,(M^*/P)\,(\omega_i - \omega_c) = 15.6\ \mu^4 \quad . \tag{B.11}$$

A more accurate calculation,[48] keeping terms of order β^2, gives 13.5 μ^4.

In region b, we have $\omega = \omega_c$, k variable, P fixed, so Eq. (5.7) gives

$$\cos\chi = (2P)^{-1}\,(k - B/k) \quad , \quad \text{and} \tag{B.12}$$

$$B = 2M^*(\omega_i - \omega_c) + p_F^2 - P^2 \quad . \tag{B.13}$$

In general, $B > 0$; for exception, see below. The maximum value of $\cos\chi$ is $\cos\chi_f$, [Eq. (B.1)]. The minimum is -1, which is reached when

$$k = k_m = (P^2 + B)^{\frac{1}{2}} - P \quad . \tag{B.14}$$

With the values previously used,

$$B = 7.2\,\mu^2, \quad k_m = 1.07\,\mu \quad . \tag{B.14a}$$

It is important that k cannot go to zero, but $k_m \ll k_i$. In this range, we use k as the independent variable, rather than x.

We have

$$k' \cos\chi' = (2P)^{-1}\,(k^2 - B) - \omega_c\beta \quad , \tag{B.15}$$

$$\omega' = \omega_c + \frac{B - k^2}{2(M + \omega_i)} \quad , \tag{B.16}$$

$$k'^2 = k^2(1 - \zeta) + B\xi \quad , \text{ and} \tag{B.17}$$

$$\zeta = \frac{\omega_c}{M + \omega_i} = 0.136 \tag{B.17a}$$

with our values of the parameters. The integral corresponding to Eq. (B.8), but over range b, is then given by

$$4P(M+\omega_i)\,\Pi = \int_{k_m}^{k_i} [(M+\omega_i)\,\omega_c + \tfrac{1}{2}B - \tfrac{1}{2}k^2] \times \times\,[k^2(1-\zeta) = B(1+\zeta) + 2P^2\zeta]\,dk^2 \quad . \tag{B.18}$$

Because $k_m^2 \ll k_i^2$ [Eq. (B.14a)], we neglect k_m^2. Neglecting some small terms, we get

$$4P(M+\omega_i)\Pi = [(M+\omega_i)\omega_c + M^*(\omega_i-\omega_c) + \tfrac{1}{2}p_F^{\,2} - \tfrac{1}{2}P^2 - \tfrac{1}{2}k_i^{\,2}] \times \times[2M^*(\omega_i-\omega_c)(1+\zeta) + p_F^{\,2}(1+\zeta) - (P^2 - k_i^{\,2})(1-\zeta)]\,. \tag{B.19}$$

In the first bracket, we use Eq. (B4b). We neglect k_i^2, p_F^2 in both brackets, and ζ and P^2 in the second bracket, and get

$$\text{II} = (M^*/2PC)\,k_i^{\,2}(\omega_i-\omega_c)\,[C\omega_c + \omega_i - \omega_c - P^2/2M^*] \,. \tag{B.20}$$

From the second bracket in Eq. (B.19) it is clear that the result may be smaller if P is large, e.g., near its maximum $p_F + k$. I and II have the same form and can be added.

$$\text{I} + \text{II} = (M^*/2P)\,k_i^{\,2}(\omega_i-\omega_c) \times \times[(2+C^{-1})\,(\omega_i - P^2/2M^*) + (1-C^{-1})\omega_c] \tag{B.21}$$

after using Eq. (B.4b). For our parameters,

$$\text{I} + \text{II} = 22.5\,\mu^4 \quad . \tag{B.22}$$

The correct result[49] happens to be also $22.5\,\mu^4$, by a cancellation of errors. See Eqs. (5.27) and (5.28) for discussion.

We now discuss two exceptional cases.

1. If P is very small, Eq. (B.1a) does not hold.

This will happen if

$$P < P_1 = -k + [k^2 + 2M^*(\omega_i - \omega_c)]^{\frac{1}{2}} \quad . \tag{B.23}$$

With our parameters, $P_1 = 1.91\,\mu$. This means that roughly 1/5 of the permitted range of P^2 is involved in Eq. (B.23).

If Eq. (B.23) holds, the entire angular range

$$-1 < \cos\chi < \cos\chi_f \tag{B.24}$$

leads to values of $\omega_f > \omega_c$; thus we have only "range a" to consider (see above). Neglecting the (rather small) effect of the center-of-mass transformation, we then need only to consider the Pauli effect. This gives, similar to Eq. (4.25),

$$L/L_o' = \tfrac{1}{2}(1+\lambda) = \tfrac{1}{2} + P/4k \quad . \tag{B.25}$$

Thus the largest value of L is obtained for $P = P_1$; it is about $0.7\,L_0'$. For $P < P_1$, Eq. (B.25) indicates a decrease of L. Likewise, for $P > P_1$,

Eq. (B.21) holds and shows that L decreases with increasing P.

2. If P is very large, then B in Eq. (B.13) becomes negative. Then Eq. (B.12) shows that cos χ can never become negative. (For B = 0, cos χ can just reach 0 at $k_m = 0$.) Instead, cos χ reaches a minimum when

$$k = k_2 = (-B)^{\frac{1}{2}} = (P^2-D^2)^{\frac{1}{2}} \quad , \tag{B.26a}$$

$$\cos\chi_{min} = (-B)^{\frac{1}{2}}/P \quad , \quad \text{and} \tag{B.26b}$$

$$D^2 = 2M^*(\omega_i-\omega_c) + p_F{}^2 \quad . \tag{B.26c}$$

For smaller k, cos χ increases again, and reaches +1 at

$$k = k_m' = P-D \tag{B.27}$$

which is the same as Eq. (B.14), except for a reversal of sign; clearly, $k_m' < k_2$. The angular range from χ_{min} to χ_i is covered twice, with two different values of k corresponding to each χ. (This is similar to the classical scattering of a heavy particle by a light target.) We believe that the integrand in Eq. (B.8) should then be changed to

$$|d\cos\chi'| \quad . \tag{B.28}$$

The contribution of the range from k_m' to k_2 is not large, and the result is not much different from Eq. (B.21).

Finally, we discuss the effect of using k'^2 instead of $3\,k'^2\cos^2\chi'$, or of $(3/2)\,k'^2\sin^2\chi'$. Equations (B.9) and (B.10a) show that the important region a involves generally small values of cos χ', hence in this region

$$\begin{aligned} 3\,k'^2\cos^2\chi' &< k'^2 \\ {}^3/_2\,k'^2\sin^2\chi' &> k'^2 \end{aligned} \quad . \tag{B.29}$$

Therefore the actual "cos" integral [Eq. (5.19)], which corresponds to J_c in Eq. (A.10a), should be smaller than the L we have calculated, whereas J_s in Eq. (A.10b) is larger. From Eqs. (B.9) and (B.7), $\cos\chi_i = 0.56$. Using Eqs. (A.11) to (A.16), it can then be seen that the effective damping is slightly larger than our L.

APPENDIX C
PAULI PRINCIPLE AND UNITARITY FOR THE MEDIUM IN THE LIPPMAN-SCHWINGER THEORY

Here we give a justification for our basic nuclear matter equations [Eq. (3.11) and (4.5)]. Although we use a specific model, the essential conclusions would presumably follow from a more general approach.

Assume that the interaction of a pion with a nucleon can be described by an equation of the form

$$T(\omega) = V(\omega) + V(\omega)\,G(\omega)\,T(\omega) \quad , \tag{C.1}$$

where $V(\omega)$ is a kernel and $G(\omega)$ is a propagator for the positive frequency components of the intermediate pion and nucleon. For the purpose of this appendix, we shall assume that the mass of the nucleon is infinite (we relax this assumption in Sec. V). We take $V(\omega)$ of the form

$$\langle\vec{q}'|V(\omega)|\vec{q}\rangle = -4\pi v(q')v(q)\left(4\omega_{q'}\omega_q\right)^{\frac{1}{2}} P_{33}(\vec{q}',\vec{q})\,\nu(q',q;\omega) \tag{C.2}$$

where $\nu(q',q;\omega)$ is a quantity which would have to be given in order to do a specific model calculation.

The particular choice of factors in Eq. (C.2) coincides with Eq. (3.4).

The pion propagator in nuclear matter is given in Eq. (2.1). This propagates both positive and negative frequency components. To obtain $G(q,\omega)$ from it, we write

$$G(k,\omega) = \frac{2\omega_k}{N(k)} \frac{1}{\omega-\omega_k} \quad , \tag{C.3}$$

where ω_k is a solution of the equation $D^{-1}(k,\omega_k) = 0$, i.e.,

$$\omega_k^2 - k^2 - \mu^2 + \Pi(k,\omega_k) = 0 \quad . \tag{C.4a}$$

In Eq. (C.4a) k is real because the intermediate integration in Eq. (C.1) ranges over a complete set of pion states; ω_k is therefore complex. The normalization $N(k)^{-1}$ in Eq. (C.3) is the residue of $D(k,\omega_k)$

$$\frac{2\omega_k}{N(k)} = \frac{2\omega_k}{2\omega_k + \frac{\partial\Pi}{\partial\Pi}(k,\omega_k)} \quad . \tag{C.4b}$$

The factor $2\omega_k$ is necessary if we want the solution of Eq. (C.1) to coincide with the Chew-Low amplitude [Eq. (3.4)].

In nuclear matter, nucleon "particles" contribute to the positive frequency propagator and "holes" to the negative frequency part. A more correct theory would therefore replace Eq. (C.3) with

$$G(k,\omega) = \frac{2\omega_k}{N(k)} \frac{Q}{\omega-\omega_k} \quad . \tag{C.5}$$

The Pauli operator Q is defined in Eq. (4.7).

We have assumed, in writing Eq. (C.3) and (C.5), that Eq. (C.4a) has only one (dominant) positive frequency solution ω_k for real values of k. If, in a particular model, this assumption should prove to be false, then Eq. (C.1) is still a correct equation, provided that we add to the right-hand side of Eqs. (C.3) and (C.5) more terms of the same form, corresponding to the additional solutions.

Let us assume that the solution to Eq. (C.1) has the form of Eq. (3.4),

$$\langle \vec{q}' | T(\omega) | \vec{q} \rangle = -4\pi v(q')v(q)\left(4\omega_q'\omega_q\right)^{-\frac{1}{2}} P_{33}(\vec{q}',\vec{q})\tau(q',q;\omega) \quad . \tag{C.6}$$

It then follows from Eqs. (C.1), (C.2), and (C.6) that

$$\tau(q',q;\omega) = v(q',k;\omega) - \frac{2}{\pi}\int_0^\infty v(q',k;\omega) \times \times \frac{k^4 dk v^2(k)}{N(k)(\omega-\omega_k)}\tau(k,q;\omega) \tag{C.7}$$

using the fact that P_{33} is normalized so that

$$\int d\Omega_k P_{33}(\vec{q}',\vec{k})\, P_{33}(\vec{k},\vec{q}) = 4\pi k^2 P_{33}(\vec{q}',\vec{q}) \, . \tag{C.8}$$

If we had used Eq. (C.5) for $G(k,\omega)$, then T would not have the simple angular dependence of Eq. (C.6).

UNITARITY

Define $\tau^+(q)$ and $\tau^-(q)$ as follows

$$\tau^{\pm}(q) \equiv \operatorname*{Lim}_{\omega\to\omega_q\pm i\varepsilon} \tau(q,q;\omega) \quad . \tag{C.9}$$

Unitarity provides a (nonlinear) relation between τ^+ and τ^- along the contour of real q. We make use of the following[82] theorem.

If T_A and T_B are defined in terms of V, G_A, and G_B,

$$T_A = V + V\, G_A T_A \tag{C.10a}$$

and

$$T_B = V + V\, G_B T_B \quad , \tag{C.10b}$$

then T_B may be expressed in terms of T_A and the difference $G_B - G_A$,

$$T_B = T_A + T_A\,(G_B - G_A)\, T_B \quad . \tag{C.10c}$$

Taking $G_B \equiv G(k,\omega_q + i\eta)$ and $G_A \equiv G(k,\omega_q - i\eta)$ with $G(k,\omega)$ given by Eq. (C.3), we find easily

$$G_B - G_A = -\frac{2\omega_k}{N(k)} (2\pi i)\, \delta(\omega_q - \omega_k) \quad . \tag{C.11}$$

Next, using the definitions in Eqs. (C.6) and (C.9), and the theorem of Eq. (C.10), we easily find the desired unitarity condition

$$\tau^+(k) - \tau^-(k) = 4ik^4 v^2(k) \left[N(k) \frac{d\omega_k}{dk} \right]^{-1} \tau^+(k)\tau^-(k) \,. \tag{C.12}$$

In the absence of the medium Eq. (C.12) reduces to the familiar condition of Eq. (3.7). But in the presence of the medium, Eq. (C.12) is just the condition we need to obtain the result in Eq. (3.11). To see this, refer to Eq. (3.10) and note

$$F_\alpha(\omega_k) = g_\alpha^+(\omega_k) - g_\alpha^-(\omega_k) \quad , \tag{C.13a}$$

where g_α^+ and g_α^- are defined in analogy to τ^+ and τ^- in Eq. (C.9). But using Eq. (3.8) and identifying $h_\alpha^\pm(\omega_k)$ with $\tau^\pm$ we find

$$F_\alpha(\omega_k) = \frac{\lambda_\alpha}{\omega_k} \left(\frac{1}{\tau^+(k)} - \frac{1}{\tau^-(k)} \right) \tag{C.13b}$$

and, using Eq. (C.12),

$$F_\alpha(\omega_k) = -4ik^4 v^2(k) \frac{\lambda_\alpha}{\omega_k} \left[N(k) \frac{d\omega_k}{dk} \right]^{-1} \quad . \tag{C.13c}$$

We may simplify Eq. (C.13c). Differentiating Eq. (C.4a), we find

$$N(k) \frac{d\omega_k}{dk} = 2k - \frac{\partial \Pi}{\partial k} (k,\omega_k) \quad . \tag{C.14}$$

Now, in our theory the dependence of $\Pi(k,\omega)$ on k is completely known [see Eqs. (2.11), and (3.31)]. Thus, we may write

$$\Pi(k,\omega) = \Pi_0(\omega) + k^2 \Pi_2(\omega) + \ldots \tag{C.15}$$

with the coefficients Π_0, Π_2, etc., known functions. The contribution of Π_0 is of course the s-wave π-nucleon scattering, Π_2 p-wave, etc.[83] We may neglect the k-dependent factor $v^2(k)$ because in realistic cases it is different from unity only for very large k. If we keep the first two terms of Eq. (C.15) and use Eq. (C.4a), then

$$\frac{\partial \Pi}{\partial k} (k,\omega_k) = 2k\Pi_2(\omega_k) = 2k \left[\frac{\omega_k^2 - k^2 - \mu^2 + \Pi_0(\omega_k)}{k^2} \right] . \tag{C.16}$$

It now follows from Eqs. (C.14) and (C.16) that

$$N(k) \frac{d\omega_k}{dk} = 2k \left[\frac{\omega_k^2 - \mu^2 + \Pi_0(\omega_k)}{k^2} \right] \tag{C.17}$$

so Eq. (C.13c) becomes

$$F_\alpha(\omega_k) = -2ik^3 v^2(k) \frac{\lambda_\alpha}{\omega_k} \left(\frac{k^2}{\omega_k^2 - \mu^2 + \Pi_0(\omega_k)} \right) . \tag{C.18}$$

A simpler but less correct expression would be to neglect the derivative $\frac{\partial \Pi}{\partial k}$ in Eq. (C.14). This is justified only for large k or small density; it leads to a result more like that of Barshay et al. We find, then,

$$N(k) \frac{d\omega_k}{dk} = 2k \qquad \text{and} \tag{C.19}$$

$$F_\alpha(\omega_k) = -2ik^3 v^2(k) \frac{\lambda_\alpha}{\omega_k} \quad . \tag{C.20}$$

For very little additional effort, computation could be carried through with the exact expressions of Eqs. (C.17) and (C.18). We have chosen to use Eqs. (C.19) and (C.20) in this work, but expect the qualitative features to be unchanged in the more exact case.

PAULI PRINCIPLE

The concern here is to find the solution $T(\omega)$ of Eq. (C.1) with $G(q,\omega)$ taken from Eq. (C.5). To solve Eq. (2.2), we need $T(\omega)$ for real ω. Of course Eq. (C.1) is valid for any ω, but to solve it we need the solution ω_q of Eq. (C.4a) because Eq. (C.1) depends on ω_q through the propagator [Eq. (C.5)]. The solution of Eq. (C.5) requires, in turn, $T(\omega_q)$. Therefore, to solve Eq. (C.1) for $T(\omega)$, we must first solve it for $T(\omega_q)$.

If we set $\omega = \omega_q$ in Eq. (C.1), then $G(k,\omega_q)$ is singular when $\omega_k = \omega_q$. Therefore it is necessary to specify the path of integration in Eq. (C.1). To obtain the proper sign for the imaginary part of ω_q, we must add $+i\eta$ to the denominator of Eq. (C.5). In this case, $G(k,\omega_q)$ may be written as a principal value and pole term,

$$G(k,\omega_q) = \frac{2\omega_k}{N(k)} P \frac{Q}{\omega_q-\omega_k} - i\pi \frac{2\omega_k}{N(k)} Q\, \delta(\omega_q-\omega_k) \quad . \tag{C.21}$$

The principal value propagator may be used to define a K-matrix

$$K(\omega_q) = V(\omega_q) + P\, V(\omega_q)\, G(\omega_q)\, K(\omega_q) \quad , \tag{C.22a}$$

and the T-matrix may be expressed in terms of the K-matrix through the help of the theorem of Eq. (C.10). We find

$$\langle \vec{q}\,' | T(\omega_q) | q \rangle = \langle \vec{q}\,' | K(\omega_q) | \vec{q} \rangle - i\pi \int \frac{d^3k}{(2\pi)^3} \langle \vec{q} | K(\omega_q) | \vec{k} \rangle \frac{2\omega_k}{N(k)} \delta(\omega_q-\omega_k)\, Q\,(\vec{P},k) \langle \vec{k} | T(\omega_q) | \vec{q} \rangle \quad . \tag{C.22b}$$

Changing the variable of integration from dk to $d\omega_k$, we find the equivalent expression

$$\langle \vec{q}\,' | T(\omega_q) | \vec{q} \rangle = \langle \vec{q}\,' | K(\omega_q) | \vec{q} \rangle - \frac{i\pi(2\omega_q)\, q^2}{N(q)\, \frac{d\omega_q}{dq}} \int \frac{d\Omega_k}{(2\pi)^3} \langle \vec{q}\,' | K(\omega_q) | \vec{k} \rangle Q(\vec{P},\vec{k}) \langle \vec{k} | T(\omega_q) | \vec{q} \rangle \quad . \tag{C.22c}$$

For $N(k)\frac{d\omega_k}{dk}$ we may use either Eq. (C.17) or Eq. (C.19).

Equation (C.22c) may be solved for $T(\omega_q)$ if $K(\omega_q)$ is known. We expect that the results of calculations will be insensitive to the nuclear matter modifications in Eq. (C.22a) and therefore that the K-matrix in Eq. (C.22c) may be taken to be the free K-matrix, $K_f(\omega)$. The reasons are (1) the high momentum intermediate states dominate Eq. (C.22a), and (2) the nuclear effects contribute only over a relatively small range of integration at low momenta. Having solved Eq. (2.22c), we can solve Eq. (C.4a) for ω_k and then Eq. (C.1) to get $T(\omega)$ for real ω.

However, there is a simpler way to proceed. Let us assume the validity of Eq. (C.22c), with K taken from the Chew-Low theory. In this case, the dependence of K on its variables is completely known. If it is true that only one mode ω_k is important, and that no singularities are crossed in moving the contour ω_k to the real axis, then $T(\omega)$ is the analytic continuation of $T(\omega_k)$, obtained by replacing ω_k everywhere in Eq. (C.22c) by ω, and k everywhere by k_ω. The result is our Eq. (4.5).

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26. Dover and Lemmer define the sign of the self-energy as indicated in Eq. (2.1), i.e., positive Π corresponds to an increase in the apparent mass of the pion, and thus to an attractive interaction. Many authors[18] use the opposite sign.
27. The "bubble" of nucleon a is commonly called a particle-hole pair; $\vec{p}_a$ is a hole in the Fermi sea, $\vec{p}_u$ is a "particle" outside the sea.
28. We shall consistently use p to denote nucleon momenta, while k, or sometimes q, denote pion momenta. We hope thereby to reduce confusion.
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31. From now on, p will denote the magnitude of the three-vector $\vec{p}$, whereas until Eq. (2.3) it denoted the four-vector $\vec{p}$, p_0.
32. It also means that, for $\rho \neq 0$, we never need consider the scattering by a nucleon of initial momentum $p = 0$, as Dover and Lemmer do in their footnote 35. This means that the threshold of pion energy remains at μ, and is not pushed up to $(\mu^2 + \frac{1}{4} p_F^2)^{\frac{1}{2}}$, as Dover and Lemmer [Eq. (3.41)] say.
33. See, e.g., Ref. 29, Fig. 8. The wave function is increased at most by 10%, usually less, at distances from 1 to 2 fm between the nucleons.
34. In terms of the diagrams in Ref. 15, this means that we must include in fourth order only 4 u (uncrossed), not 4 c (see our Fig. 1). In the "crossed" diagram 4 c, two pions are emitted before the incident π is absorbed. Such a process is explicitly excluded in the one-meson approximation. On the other hand, diagrams like $f_u^{(6)}$ of Dover and Lemmer are permitted.

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36. In the simplest case of the one-meson approximation, f is proportional to $\sin\delta\, e^{i\delta}$ whose imaginary part is $\sin^2\delta > 0$.

37. We labored for a long time before coming to this conclusion. We are very grateful to D. G. Boulware and S. Blaha, of the theoretical physics seminar at the University of Washington, for making a suggestion (in June 1973) which led to this solution of the unitarity problem.

38. The unitarity relation for the medium is discussed in Appendix C, and it is found that the correct form is model dependent. We have decided to take a form as close as possible to the free space solution for the sake of simplicity, believing that the details of the theory will not be greatly influenced by the choice.

39. Chew and Low take the pion mass to be unity; we take it to be μ, hence the extra factor μ^2 in Eq. (3.7). We find it useful to be able to check dimensions. However, $\hbar = c = 1$ in this report, as in Ref. 16.

40. As Dover and Lemmer pointed out (their Fig. 11), this cut-off is in conflict with that which one might wish to use to reproduce the shape of the 33-resonance. On the basis of that shape, A. A. Carter, J. R. Williams, D. V. Bugg, P. J. Bussey, and D. R. Dance, Nucl. Phys. B26, 445 (1971), propose a form factor

$$U(k) = (1 + k^2a^2)^{-1} \qquad (a)$$

$$a = 0.628\ \mu^{-1} = 0.90\ \text{fm}\ , \qquad (b)$$

i.e., a very long-range cut-off which has been used by a number of theorists working on π-nucleon processes. However, this long-range cut-off is not justifiable by high-energy theory as has been shown by J. Hamilton and collaborators. They show that the lowest-energy diagram which can modify the πNN vertex is that shown in our Fig. 6. It effectively involves the transmission of three pions between nucleon and pion, therefore certainly

$$a \leq 0.333\ \mu^{-1} = 0.48\ \text{fm}\ . \qquad (c)$$

Investigating this diagram in more detail, J. Hamilton, "On Zero Mass Pions," Nordita (Copenhagen) 1969, p. 57-60, and H. J. Braathen, Nucl. Phys. B44, 93 (1972) find that the mass transmitted in our Fig. 6 should be at least $\mu\sqrt{15}$. However, the vertex function could be

$$U(k) = (1 + k^2/15\mu^2)^{-1} \qquad (d)$$

rather than (a), and this corresponds roughly to the choice (c) if the form (a) is used. Hamilton and Braathen emphasize the (d) is a maximum estimate for the nonlocality of the πNN vertex a could be much smaller. The shape of the observed relation between 33-phase shift and energy, above the resonance, must then be attributed to other causes, such as the effect of other resonances. On the other hand, (c) or (d) do not give as high a cut-off as $k = 12$ to 14 μ. This is partly explained by the fact (Ref. 35) that part of the strength of the 33-resonance comes from the σ-meson exchange (Fig. 4). In the framework of the pure Chew-Low theory, it seems to us justified to use the high cut-off at $k = 12$ to 14 μ.

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44. The coefficient is chosen to give the simple result given in Eqs. (4.9) and (A.12a).

45. In principle, it is possible to use the pion-nucleon resonance energy in the center-of-mass frame of these two particles. But then, for the complex nucleus, we should use the center of mass of the pion and a nucleon at rest inside the nucleus, a highly artificial procedure which would not change the final result. In any case, it would be incorrect to compare the resonance energy in the center-of-mass frame of pion and complex nucleus (which is essentially the laboratory frame) to the resonance in the center of mass of pion and free nucleon.

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$$M^*/M \approx 0.70 \qquad (5.2a)$$

$$C_m = 1.43\ .$$

Therefore it should be better for our purposes to use

$$C_m = 1 + 0.24\, p_F^2\ . \qquad (5.2b)$$

According to Jeukenne et al., Fig. 5, Eq. (5.2a) appears to be in agreement with experiment at least up to the point where the potential energy is zero, which occurs at

$$E(p) = \frac{-U_0}{C_m - 1} \approx 160\ \text{MeV}\ . \qquad (5.2c)$$

This is amply sufficient to cover the range of recoil energies needed in our theory. For lower density, E(p) is slightly higher than Eq. (5.2c). (It should be noted that Jeukenne et al. take $p_F = 1.40\ \text{fm}^{-1}$, whereas we consider $1.33\ \text{fm}^{-1}$ to be correct, but the value of $C_m = 1.43$ comes directly from experiment.)

47. Trouble arises here from the fact that, for real values of k, the energy $\omega(k)$ is complex, as is ω_i. The imaginary part of $\omega_i - \omega(k)$ will, in general, not be zero, whereas all other terms in Eq. (5.7) are real, so Eq. (5.7) cannot be strictly fulfilled. So we make an approximation, which we hope is good because $\mathrm{Im}(\omega_k)$ is not very large in practice. It is probably best to postulate that Eq. (5.7) holds for the real part of $\omega_i - \omega(k)$. This is the only point in the calculation with real k where the complex nature of ω requires an approximation.

48. In our own calculations, we have carried terms of order β^2, which give corrections of order 10%. We will indicate the exact results, including these terms, in App. B.

49. Making a Lorentz transformation from the cm to the laboratory system gives $W = \gamma W'$ since $P' = 0$ by definition.

50. It is not obvious that

$$\cos \chi < \cos \chi_i = \frac{p^2 + k_i^2 - p_F^2}{2pk_i}$$

is always true. But for most of the conceivable reasonable relations between ω and k, it will be.

51. In general, the second term in Eq. (5.20) is smaller than the first. Taking only the first term in Eq. (5.20), and assuming ω' to be constant, Eq. (5.19) integrates to

$$1/3\ \omega'[(k_i \cos \chi_i - \omega_i\beta)^3 + (k_m + \omega_m\beta)^3], \qquad (5.19a)$$

where k_m, ω_m are the minimum values of k, ω corresponding to $\cos \chi' = -1$. The second term in Eq. (5.19a) will, in general, be much smaller than the first.

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59. As we pointed out in Sec. VI, that theory is justified to all orders in ρ at low pion energy, because then the damping term is zero.

60. This was clearly recognized by Silbar and Sternheim.[12]

61. The approximation may underestimate the integral in Eq. (9.4) by about 5%.

62. The true absorption of pions, by the process in Eq. (1.11), is negligible in comparison. According to Eq. (6.1), we must add to k^2 a term $i\, C_0\rho^2k^2$, so that k^2 gets multiplied by $1 + i\, C_0\rho^2$. Thus the change is

$$\Delta(\mathrm{Im}\ k) = \tfrac{1}{2}\, C_0\rho^2(\mathrm{Re}\ k) \quad . \qquad (a)$$

For $\rho = \frac{1}{4}\rho_0$, our most important density, this amounts to about 0.001 μ, compared with about 0.2 μ from the analog of Fig. 10. For $\rho = \frac{1}{2}\rho_0$ and $k_0/\mu = 1.25$, we get $\Delta(\mathrm{Im}\ k/\mu) = 0.004$, still negligible compared with 0.25 from Fig. 10.

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75. The effect of L + 1 and L + 2 might be included by Born approximation; perhaps also that of L itself.

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